

Predicate Calculus - Semantics 1/4

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Introduction to predicate calculus (1/2)

- Propositional logic (sentence logic) dealt quite satisfactorily with sentences using conjunctive words (접속사) like **not**, **and**, **or**, and **if ... then**. But it fails to reflect the *finer logical structure* of the sentence
- What can we reason about a sentence itself which deals with its **target domain**?
 - ex. Jane is taller than Alice (target domain : human being)
 - ex2. For natural numbers x and y , $x+y \geq -(x+y)$ (target domain: \mathcal{N})
- What can we reason about a sentence itself which also deal with modifiers like **there exists...**, **all ...**, **among ...** and **only** ?
 - Note that these modifiers enable us to reason about an **infinite** domain because we do not have to enumerate all elements in the domain

Introduction to predicate calculus (2/2)

- Ex. Every student is younger than some instructor
 - We could simply identify this assertion with a propositional atom p .
However, this fails to reflect the finer logical structure of this sentence
- This statement is about **being a student**, **being an instructor** and **being younger than somebody else** for a set of university members as a target domain
 - We need to express them and use **predicates** for this purpose
 - $S(\text{yunho})$, $I(\text{moonzoo})$, $Y(\text{yunho}, \text{moonzoo})$
- We need variables x , y to not to write down all instance of $S(-)$, $I(-)$, $Y(-)$
 - Every student x is young than some instructor y
- Finally, we need **quantifiers** to capture the actual elements by variables
 - For every x , if x is a student, then there is some y which is an instructor such that x is younger than y
 - $\forall x (S(x) \rightarrow (\exists y (I(y) \wedge Y(x,y))))$

Relations and predicates

- The axioms and theorems of mathematics are defined on **arbitrary sets (domain)** such as the set of integers \mathbb{Z}
 - ex. Fermat's last theorem
 - If an integer n is greater than 2, then the equation $a^n + b^n = c^n$ has no solutions in non-zero integers a , b , and c .
 - Can we express the Fermat's last theorem in propositional logic?
- The **predicate calculus** extends the propositional calculus with **predicate** letters that are interpreted as **relations on a domain**
 - i.e., predicates are interpreted upon domain
- Def 5.2. A relation can be represented by a boolean valued function $R:D^n \rightarrow \{T,F\}$, by mapping an n -tuple to T iff it is included in the relation
 - $R(d_1, \dots, d_n) = T$ iff $(d_1, \dots, d_n) \in \mathcal{R}$

Predicate formulas

- Let \mathcal{P} , \mathcal{A} and \mathcal{V} be countable sets of symbols called **predicate letters**, **constants**, and **variables**, respectively.

- $\mathcal{P}=\{p,q,r\}$ $\mathcal{A}=\{a,b,c\}$, $\mathcal{V}=\{x,y,z\}$

- Def 5.4 Atomic formulas and formulas

- atomic formula

- argument ::= x for any $x \in \mathcal{V}$
- argument ::= a for any $a \in \mathcal{A}$
- argument_list ::= argument⁺
- atomic_formula ::= p |
 $p(\text{argument_list})$ for any $p \in \mathcal{P}$

- formula ::= atomic_formula
- formula ::= \neg formula
- formula ::= formula \vee formula
- formula ::= $\forall x$ formula
- formula ::= $\exists x$ formula

- $\forall x \forall y (p(x, y) \rightarrow p(y, x)).$

- $\forall x \exists y p(x, y).$

- $\exists x \exists y (p(x) \wedge \neg p(y)).$

- $\forall x p(a, x).$

- $\forall x (p(x) \wedge q(x)) \leftrightarrow (\forall x p(x) \wedge \forall x q(x)).$

- $\exists x (p(x) \vee q(x)) \leftrightarrow (\exists x p(x) \vee \exists x q(x)).$

- $\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x)).$

- $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x)).$

Free and bound variables

- Def 5.6
 - \forall is the **universal quantifier** and is read ‘for all’.
 - \exists is the **existential quantifier** and is read ‘there exists’.
 - In a quantified formula $\forall xA$, x is the **quantified variable** and A is the **scope** of the quantified variable.
- Def 5.7 Let A be a formula. An occurrence of a variable x in A is a **free variable** of A iff x is not within the scope of a quantified variable x .
 - Notation: $A(x_1, \dots, x_n)$ indicates that the set of free variables of the formula A is a subset of $\{x_1, \dots, x_n\}$. A variable which is not free is **bound**.
 - If a formula has no free variable it is **closed**
 - If $\{x_1, \dots, x_n\}$ are all the free variables of A , the **universal closure** of A is $\forall x_1 \dots \forall x_n A$ and the **existential closure** is $\exists x_1 \dots \exists x_n A$
- Ex 5.8 $p(x,y), \exists y p(x,y), \forall x \exists y p(x,y)$
- Ex 5.9
 - In $(\forall x p(x)) \wedge q(x)$, the occurrence of x in $p(x)$ is bound and the occurrence in $q(x)$ is free. The universal closure is $\forall x (\forall x p(x) \wedge q(x))$.
 - Obviously, it would have been better to write the formula as $\forall x p(x) \wedge q(y)$ where y is the free variable

Interpretations (1/5)

- Def 5.10 Let U be a set of formulas s.t. $\{p_1, \dots, p_m\}$ are all the predicate letters and $\{a_1, \dots, a_k\}$ are all the constant symbols appearing in U . An **interpretation** \mathcal{I} is a triple $(D, \{R_1, \dots, R_m\}, \{d_1, \dots, d_k\})$, where
 - D is a **non-empty** set,
 - R_i is an n_i -ary relation on D that is assigned to the n_i -ary predicate p_i
 - Notation: $p_i^{\mathcal{I}} = R_i$
 - $d_i \in D$ is an element of D that is assigned to the constant a_i
 - Notation: $a_i^{\mathcal{I}} = d_i$
- Ex 5.11. Three numerical interpretations for $\forall x p(a, x)$:
 - $\mathcal{I}_1 = (\mathcal{N}, \{\leq\}, \{0\})$, $\mathcal{I}_2 = (\mathcal{N}, \{\leq\}, \{1\})$. $\mathcal{I}_3 = (\mathcal{Z}, \{\leq\}, \{0\})$.
 - $\mathcal{I}_4 = (\mathcal{S}, \{\text{substr}\}, \{\text{""}\})$ where \mathcal{S} is the set of strings on some alphabet

Interpretations (2/5)

- Def 5.12 Let \mathcal{I} be an interpretation. An assignment $\sigma_{\mathcal{I}}: \mathcal{V} \rightarrow \mathbf{D}$ is a function which maps every variable to an element of the domain of \mathcal{I} . $\sigma_{\mathcal{I}}[x_i \leftarrow d_i]$ is an assignment that is the same as $\sigma_{\mathcal{I}}$ except that x_i is mapped to d_i
- Def 5.13 Let A be a formula, \mathcal{I} an interpretation and $\sigma_{\mathcal{I}}$ an assignment. $v_{\sigma_{\mathcal{I}}}(A)$, the **truth value** of A **under** $\sigma_{\mathcal{I}}$ is defined by induction on the structure of A :
 - Let $A = p_k(c_1, \dots, c_n)$ be an atomic formula where each c_i is either a variable x_i or a constant a_i . $v_{\sigma_{\mathcal{I}}}(A) = \mathbf{T}$ iff
 - $\langle d_1, \dots, d_n \rangle \in R_k$ where R_k is the relation assigned by \mathcal{I} to p_k and
 - d_i is the domain element assigned to c_i , either
 - by \mathcal{I} if c_i is a constant or
 - by $\sigma_{\mathcal{I}}$ if c_i is variable
 - $v_{\sigma_{\mathcal{I}}}(\neg A) = \mathbf{T}$ iff $v_{\sigma_{\mathcal{I}}}(A) = \mathbf{F}$
 - $v_{\sigma_{\mathcal{I}}}(A_1 \vee A_2) = \mathbf{T}$ iff $v_{\sigma_{\mathcal{I}}}(A_1) = \mathbf{T}$ or $v_{\sigma_{\mathcal{I}}}(A_2) = \mathbf{T}$
 - $v_{\sigma_{\mathcal{I}}}(\forall x A_1) = \mathbf{T}$ iff $v_{\sigma_{\mathcal{I}}[x \leftarrow d]}(A_1) = \mathbf{T}$ for **all** $d \in \mathbf{D}$
 - $v_{\sigma_{\mathcal{I}}}(\exists x A_1) = \mathbf{T}$ iff $v_{\sigma_{\mathcal{I}}[x \leftarrow d]}(A_1) = \mathbf{T}$ for **some** $d \in \mathbf{D}$



Interpretations (3/5)

- Thm 5.14 Let A be a closed formula. Then $v_{\sigma_{\mathcal{I}}}(A)$ does **not** depend on $\sigma_{\mathcal{I}}$. In such cases, we use simply $v_{\mathcal{I}}(A)$ instead of $v_{\sigma_{\mathcal{I}}}(A)$
- (important!) Thm 5.15 Let $A' = A(x_1, \dots, x_n)$ be a non-closed formula and let \mathcal{I} be an interpretation. Then:
 - $v_{\sigma_{\mathcal{I}}}(A') = T$ for **some** assignment $\sigma_{\mathcal{I}}$ iff $v_{\mathcal{I}}(\exists x_1 \dots \exists x_n A') = T$
 - $v_{\sigma_{\mathcal{I}}}(A') = T$ for **all** assignment $\sigma_{\mathcal{I}}$ iff $v_{\mathcal{I}}(\forall x_1 \dots \forall x_n A') = T$
 - Thm 5.15 is important since we have many chances to add or remove quantified variables to and from formula during proofs.
- Def 5.16 A closed formula A is true in \mathcal{I} or \mathcal{I} is a model for A , if $v_{\mathcal{I}}(A) = T$.
 - Notation: $\mathcal{I} \models A$
 - Note that we overload \models with usual logical consequence as in propositional logic
 - $\{A_1, A_2, A_3\} \models A$
- Def 5.18 A closed formula A is **satisfiable** if for **some** interpretation \mathcal{I} , $\mathcal{I} \models A$. A is **valid** if for **all** interpretations \mathcal{I} , $\mathcal{I} \models A$
 - Notation: $\models A$.

Interpretation (4/5)

- Ex 5.19 $\models (\forall x p(x)) \rightarrow p(a)$
 - Suppose that it is **not**. Then there must be an interpretation $\mathcal{I} = (D, \{R\}, \{d\})$ such that $v_{\mathcal{I}}(\forall x p(x)) = T$ and $v_{\mathcal{I}}(p(a)) = F$
 - By Thm 5.15, $v_{\sigma_{\mathcal{I}}}(p(x)) = T$ for all assignments $\sigma_{\mathcal{I}}$, in particular for the assignment $\sigma'_{\mathcal{I}}$ that assigns d to x (i.e. $v_{\sigma'_{\mathcal{I}}}(p(x)) = T$). But $p(a)$ is closed, so $v_{\sigma'_{\mathcal{I}}}(p(a)) = v_{\mathcal{I}}(p(a)) = F$, a **contradiction**

Example 5.20 Here is a semantic analysis of the formulas from Example 5.5:

- $\forall x \forall y (p(x, y) \rightarrow p(y, x))$

The formula is satisfiable in an interpretation where p is assigned a symmetric relation like $=$.
- $\forall x \exists y p(x, y)$

The formula is satisfiable in an interpretation where p is assigned a relation that is a total function, such as $(x, y) \in R$ iff $y = x + 1$ for $x, y \in \mathcal{Z}$.
- $\exists x \exists y (p(x) \wedge \neg p(y))$

This formula is satisfiable only in a domain with at least two elements.

Interpretation (5/5)

- $\forall x p(a, x)$

This expresses the existence of a special element. For example, if p is interpreted by the relation \leq on the domain \mathcal{N} , then the formula is true for $a = 0$. If we change the domain to \mathcal{Z} the formula is false for the same assignment of \leq to p . Thus a change of domain alone can falsify a formula.

- $\forall x(p(x) \wedge q(x)) \leftrightarrow (\forall x p(x) \wedge \forall x q(x))$

The formula is valid. We prove the forward direction and leave the converse as an exercise. Let $\mathcal{I} = (D, \{R_1, R_2\}, \{ \})$ be an arbitrary interpretation. By Theorem 5.15, $v_{\sigma_{\mathcal{I}}}(p(x) \wedge q(x)) = T$ for all all assignments $\sigma_{\mathcal{I}}$, and by the inductive definition of an interpretation, $v_{\sigma_{\mathcal{I}}}(p(x)) = T$ and $v_{\sigma_{\mathcal{I}}}(q(x)) = T$ for all assignments $\sigma_{\mathcal{I}}$. Again by Theorem 5.15, $v_{\mathcal{I}}(\forall x p(x)) = T$ and $v_{\mathcal{I}}(\forall x q(x)) = T$, and by the definition of interpretation $v_{\mathcal{I}}(\forall x p(x) \wedge \forall x q(x)) = T$.

Show that \forall does not distribute over disjunction by constructing a falsifying interpretation for $\forall x(p(x) \vee q(x)) \leftrightarrow (\forall x p(x) \vee \forall x q(x))$.

- $\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$

This is a valid formula, but its converse is not.