Predicate Calculus
- Semantics 2/4

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Logical equivalence (1/3)

- Def 5.21 Given two closed formulas A_1, A_2, if $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretation $\mathcal{I}$, then A_1 is logically equivalent to A_2
  - Notation: $A_1 \equiv A_2$
- Let A be a closed formula and U a set of closed formulas. If for all interpretations $\mathcal{I}$, $v_{\mathcal{I}}(A) = T$ whenever $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$, then A is a logical consequence of U
  - Notation: $U \vDash A$
- Thm 5.22 $A \equiv B$ iff $\vDash A \leftrightarrow B$, $U \vDash A$ iff $\vDash (A_1 \wedge \ldots \wedge A_n) \rightarrow A$

Logical equivalence (2/3)

$\forall x A(x) \leftrightarrow \neg \exists x \neg A(x)$	$\exists x A(x) \leftrightarrow \neg \forall x \neg A(x)$	**simple but important duality**
$\forall x \forall y A(x,y) \leftrightarrow \forall y \forall x A(x,y)$	$\exists x \exists y A(x,y) \leftrightarrow \exists y \exists x A(x,y)$	
$\exists x \forall y A(x,y) \rightarrow \forall y \exists x A(x,y)$		
$(\exists x A(x) \vee B) \leftrightarrow \exists x (A(x) \vee B)$	$(\forall x A(x) \vee B) \leftrightarrow \forall x (A(x) \vee B)$	**passing quantifier over closed formula**
$(B \vee \exists x A(x)) \leftrightarrow \exists x (B \vee A(x))$	$(B \vee \forall x A(x)) \leftrightarrow \forall x (B \vee A(x))$	
$(\exists x A(x) \wedge B) \leftrightarrow \exists x (A(x) \wedge B)$	$(\forall x A(x) \wedge B) \leftrightarrow \forall x (A(x) \wedge B)$	
$(B \wedge \exists x A(x)) \leftrightarrow \exists x (B \wedge A(x))$	$(B \wedge \forall x A(x)) \leftrightarrow \forall x (B \wedge A(x))$	
$\forall x (A \rightarrow B(x)) \leftrightarrow (A \rightarrow \forall x B(x))$	$\forall x (A(x) \rightarrow B) \leftrightarrow (\exists x A(x) \rightarrow B)$	
$(\exists x (A(x) \vee B(x)) \leftrightarrow (\exists x A(x) \vee \exists x B(x))$	$\forall x (A(x) \wedge B(x)) \leftrightarrow (\forall x A(x) \wedge \forall x B(x))$	**passing quantifiers through $\wedge$ and $\vee$** (covers all rows from $(\exists x A(x) \vee B)$ down to here)
$\forall x A(x) \vee \forall x B(x)) \rightarrow \forall x (A(x) \vee B(x))$	$\exists x (A(x) \wedge B(x)) \rightarrow (\exists x A(x) \wedge \exists x B(x))$	
$\forall x (A(x) \leftrightarrow B(x)) \rightarrow (\forall x A(x) \leftrightarrow \forall x B(x))$	$\forall x (A(x) \leftrightarrow B(x)) \rightarrow (\exists x A(x) \leftrightarrow \exists x B(x))$	**passing quantifiers through $\rightarrow$**
$\exists x (A(x) \rightarrow B(x)) \leftrightarrow (\forall x A(x) \rightarrow \exists x B(x))$	$(\exists x A(x) \rightarrow \forall x B(x)) \rightarrow \forall x (A(x) \rightarrow B(x))$	
$\forall x (A(x) \vee B(x)) \rightarrow (\forall x A(x) \vee \exists x B(x))$	$\forall x (A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \forall x B(x))$	
$\forall x (A(x) \rightarrow B(x)) \rightarrow (\exists x A(x) \rightarrow \exists x B(x))$	$\forall x (A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \exists x B(x))$	

Logical equivalence (3/3)

- Find counter examples for converse way
  - $(\forall x A(x) \lor \forall x B(x)) \rightarrow \forall x(A(x) \lor B(x))$
  - $\exists x\ (A(x) \land B(x)) \rightarrow (\exists x\ A(x) \land \exists x\ B(x))$
- Passing quantifiers through implication
  - $\exists x(A(x) \rightarrow B(x)) \equiv \exists x\ (\neg A(x) \lor B(x)) \equiv \exists x \neg A(x) \lor \exists x B(x)$

  $\equiv \neg\exists x\ \neg A(x) \rightarrow \exists x B(x) \equiv \forall x\ A(x) \rightarrow \exists x\ B(x)$
- Ex 5.23 $\models \forall x\ (A(x) \lor B(x)) \rightarrow (\forall x\ A(x) \lor \exists x\ B(x))$
  - $\forall x\ (A(x) \lor B(x)) \rightarrow (\forall x\ A(x) \lor \exists x\ B(x)) \equiv$

    $\forall x\ (A(x) \lor B(x)) \rightarrow (\exists x \neg A(x) \rightarrow \exists x\ B(x)) \equiv$

    $\exists x \neg A(x) \rightarrow (\forall x(A(x) \lor B(x)) \rightarrow \exists x B(x))$

    (note. $A \rightarrow (B \rightarrow C) \equiv B \rightarrow (A \rightarrow C)$)
  - If $v_{\mathcal{I}}(\exists x \neg A(x)) = F$ or $v_{\mathcal{I}}(\forall x\ (A(x) \lor B(x))) = F$, the formula is true.
  - Thus, we need only show $v_{\mathcal{I}}(\exists x\ B(x))=T$ whenever $v_{\mathcal{I}}(\exists x \neg A(x))=T$ and $v_{\mathcal{I}}(\forall x(A(x) \lor B(x)))=T$
  - By Thm 5.15, for some assignment $\sigma'_{\mathcal{I}}$, $v_{\sigma'_{\mathcal{I}}}(\neg A(x))=T$ and thus $v_{\sigma'_{\mathcal{I}}}(A(x))=F$. Using Thm 5.15 again, $v_{\sigma_{\mathcal{I}}}(A(x) \lor B(x)) = T$ under all assignments, in particular under $\sigma'_{\mathcal{I}}$. Thus, $v_{\sigma'_{\mathcal{I}}}(B(x)) = T$, and using Thm 5.15 yet again, $v_{\mathcal{I}}(\exists x\ B(x)) = T$.

Necessity of functions

- Notation. $\forall xyz \equiv \forall x \forall y \forall z$, $\exists\ xyz \equiv \exists x \exists y \exists z$
- One can always do without function symbols by using predicate symbols instead
  - note that a function is defined as a relation as a predicate is
- Andy and Paul have the same maternal grandmother (외할머니)
  - $\forall xyuv\ (M(x,y) \land M(y,Andy) \land M(u,v) \land M(v,Paul) \rightarrow x=u)$.
- The function symbols of predicate logic give us a ways of avoiding this ugly encoding, for they allow us to represent y's mother in a more direct way.
  - Instead of writing M(x,y) to mean that x is y's mother, we simply write m(y) to mean y's mother where m is a function symbol
  - m(m(Andy)) = m(m(Paul))

Introduction of function symbols (Sect 7.1)

- Ex 7.1 $(x>y) \rightarrow ((x+1) > (y+1))$ can be written in prefix notation as $>(x,y) \rightarrow >(+(x,1),+(y,1))$. It is interpreted instance of the following formula in the predicate calculus: $p(x,y) \rightarrow p(f(x,a),f(y,a))$ where
  - > is assigned to p, + is assigned to f and 1 to a
- Def 7.2 Let $\mathcal{F}$ be a countable set of function symbols. The following grammar rules define terms, a generalization of constants and variables. The rule for atomic_formula is modified to take a term_list as its argument
  - atomic formula
  - term ::= x for any $x \in \mathcal{V}$
  - term ::= a for any $a \in \mathcal{A}$
  - term ::= f(term_list) for any $f \in \mathcal{F}$
  - term_list ::= $term^+$
  - atomic_formula ::= p (term_list) for any $p \in \mathcal{P}$
- As with predicate symbols, function symbols have a fixed arity
  - functions are denoted by {f,g,h}

Functions (1/2)

- Ex 7.3
  - terms: a, x, f(a,x), f(g(x),y), g(f(a,g(b))
  - atomic formulas: p(a,b), p(x,f(a,x)), q(f(a,a),f(g(x),g(x))
- Def 7.4 A term or atom is ground iff it contains no variables. A formula is ground iff it contains no quantifiers and no variables. A formula A is a ground instance of a quantifier-free formula A iff it can be obtained from A by substituting ground terms for the (free) variables in A
- Def 7.5 Let U be a set of formulas s.t. $\{p_1,\ldots,p_k\}$ are all the predicate symbols, $\{f_1,\ldots f_l\}$ are all the function symbols and $\{a_1,\ldots,a_m\}$ are all the constant symbols appearing in U. An interpretation $\mathcal{I}$ is a 4-tuple
  - $(D, \{R_1,\ldots,R_k\}, \{F_1,\ldots,F_l\}, \{d_1,\ldots d_m\})$
    - D is a non-empty set
    - an assignment of n_i-ary relations R_i on D to the n_i-ary predicate symbols p_i
    - an assignment of n_i-ary functions F_i on D to the n_i-ary function symbols f_i
      - Notation: $f_i^{\mathcal{I}} = F_i$
    - an assignment of elements $d_i \in D$ to the constant symbols a_i

Functions (2/2)

- Def 7.6 Given a ground term t, $v_{\mathcal{I}}(t)$, the value of the term in the interpretation $\mathcal{I}$, is defined by induction:
  - $v_{\mathcal{I}}(a_i) = d_i$
  - $v_{\mathcal{I}}(f_i(t_1,\ldots,t_n)) = F_i(v_I(t_1),\ldots,v_{\mathcal{I}}(t_n))$

  $v_{\mathcal{I}}(A)$, the value of a formula, is also defined by induction. For atomic formulas:
  - $v_{\mathcal{I}}(p_i(t_1,\ldots,t_n)) = T$ iff $(v_I(t_1),\ldots,v_{\mathcal{I}}(t_n)) \in R$
- Ex 7.7 $(\mathcal{Z}, \{\leq\},+,\{1\}) \vDash \forall x\, \forall y\, (p(x,y) \rightarrow p(f(x,a),f(y,a)))$
  - i.e., $\forall x \forall y\, ((x \leq y) \rightarrow (x+1 \leq y+1))$
  - However, the formula is not valid since it is falsified by the interpretation $(\mathcal{Z}, \{\leq\},*,\{-1\})$
    - $4 \leq 5$ but $4*-1 \not\leq 5*-1$