Predicate Calculus
- Natural deduction (1/2)

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Natural deduction

- Proofs in the natural deduction for predicate logic are similar to those for propositional logic
  - We have new proof rules for dealing with $\forall$, $\exists$ and with the equality (=) symbol
  - As in the natural deduction for propositional logic, the additional rules for the quantifiers and equality will come in two flavors
    - introduction and elimination rules

Summary of proof rules of natural deduction

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1$ $\quad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1$ $\quad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	assumption: $\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	assumption: $\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$
$\neg$	assumption: $\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

$$\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad \dfrac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

assumption:

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad \dfrac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

Example 1

$$p \land \neg q \to r, \neg r, p \vdash q$$

1	$p \land \neg q \to r$	premise
2	$\neg r$	premise
3	p	premise
4	$\neg q$	assumption
5	$p \land \neg q$	$\land$i 3, 4
6	r	$\to$e 1, 5
7	$\bot$	$\neg$e 6, 2
8	$\neg\neg q$	$\neg$i 4–7
9	q	$\neg\neg$e 8

Example 2

$$p \to q \vdash \neg p \lor q$$

1	$p \to q$	premise
2	$\neg p \lor p$	LEM
3	$\neg p$	assumption
4	$\neg p \lor q$	$\lor i_1$ 3
5	p	assumption
6	q	$\to$e 1, 5
7	$\neg p \lor q$	$\lor i_2$ 6
8	$\neg p \lor q$	$\lor$e 2, 3–4, 5–7

Example 3 (Law of Excluded Middle)

$$\frac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee i_1$ 2
4	$\bot$	$\neg e$ 3, 1
5	$\neg\phi$	$\neg i$ 2–4
6	$\phi \vee \neg\phi$	$\vee i_2$ 5
7	$\bot$	$\neg e$ 6, 1
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg i$ 1–7
9	$\phi \vee \neg\phi$	$\neg\neg e$ 8

The proof rules for ∀ and ∃

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \phi[x_0/x] \end{array}}}{\forall x \phi}\ \forall x\, i$$

$$\frac{\forall x \phi}{\phi[t/x]}\ \forall x\, e$$

$$\frac{\phi[t/x]}{\exists x \phi}\ \exists x\, i$$

$$\frac{\exists x\, \phi \qquad \boxed{\begin{array}{ll} x_0 & \phi[x_0/x] \\ & \quad\vdots \\ & \quad\chi \end{array}}}{\chi}\ \exists x e$$

$\forall$x e, $\forall$x i

- $\forall$x e
  - If $\forall x\ \phi$ is true, then you could replace the x in ϕ by any term t
    - t must be free for x in ϕ
  - Ex. Let $\phi = \exists y\ (x < y)$
    - Suppose that we replace x with y, i.e., $\phi[y/x] = \exists\ y\ (y < y)$
    - very different meaning!

$$\frac{\forall x \phi}{\phi[t/x]}\ \forall x\ e$$

- $\forall$x i
  - If, starting with a 'fresh' variable x_0, you are able to prove some formula $\phi[x_0/x]$ with x_0 in it, then (because x_0 is fresh) you can derive $\forall x\ \phi$
  - x_o does not occur outside the box

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \phi[x_0/x] \end{array}}}{\forall x \phi}\ \forall x\ i$$

Example

- $\forall x\,(P(x) \rightarrow Q(x)),\ \forall x\,P(x) \vdash \forall x\,Q(x)$

①		$\forall x\,(P(x) \rightarrow Q(x))$	Premise
②		$\forall x\,P(x)$	Premise
③	x_0	$P(x_0) \rightarrow Q(x_0)$	$\forall x$ e 1
④		$P(x_0)$	$\forall x$ e 2
⑤		$Q(x_0)$	$\rightarrow$ e 3,4
⑥		$\forall x\,Q(x)$	$\forall x$ i 3-5

$\exists$x i, $\exists$x e

- $\exists$ x i
  - It simply says that we can deduce $\exists$ x ϕ whenever we have ϕ[t/x] for some term t
    - t must be free for x in ϕ

$$\frac{\phi[t/x]}{\exists x\, \phi}\ \exists x\ i$$

- $\exists x\ e$
  - We know $\exists$ x ϕ is true, so ϕ is true for at least one value of x
    - So we do a case analysis over all those possible values, writing x_0 as a generic value representing them all

$$\frac{\exists x\, \phi \qquad \boxed{\begin{array}{ll} x_0 & \phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists x e$$

Example

- $\forall x\,(P(x) \rightarrow Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$

①		$\forall x\,(P(x) \rightarrow Q(x))$	Premise
②		$\exists x\,P(x)$	Premise
③	x_0	$P(x_0)$	Assumption
④		$P(x_0) \rightarrow Q(x_0)$	$\forall x$ e 1
⑤		$Q(x_0)$	$\rightarrow$e 4,3
⑥		$\exists x\,Q(x)$	$\exists x$ i 5
⑦		$\exists x\,Q(x)$	$\exists x$ e 2,3-6