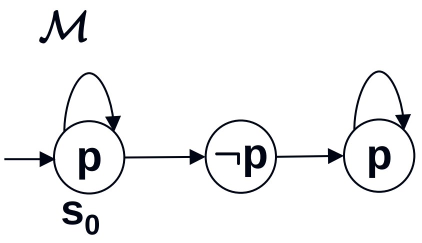
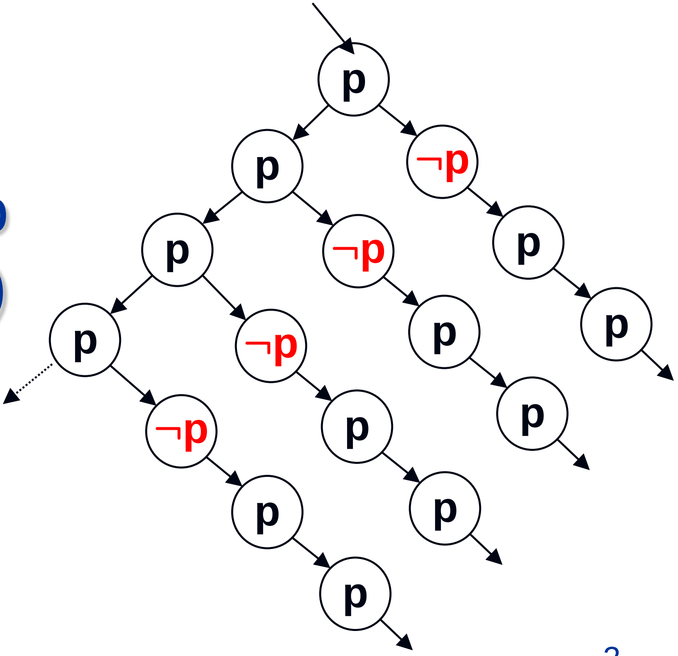
Temporal Logic
- LTL, CTL, and CTL*

Moonzoo Kim
CS Dept. KAIST

CTL is not more expressive than LTL

- CTL cannot select a range of paths
  - F G p in LTL is not equivalent to AF AG p
    - $\mathcal{M}, s_0 \vDash$ F G p but $\mathcal{M}, s_0 \nvDash$ AF AG p
    - AF AG p is strictly stronger than F G p
    - AF EG p is strictly weaker than F G p
- Similarly, F p $\rightarrow$ F q is not equivalent to AF p $\rightarrow$ AF q, neither to AG (p $\rightarrow$ AF q)
- Remark
  - F X p ≡ X F p in LTL
  - AF AX p is not equivalent to AX AF p

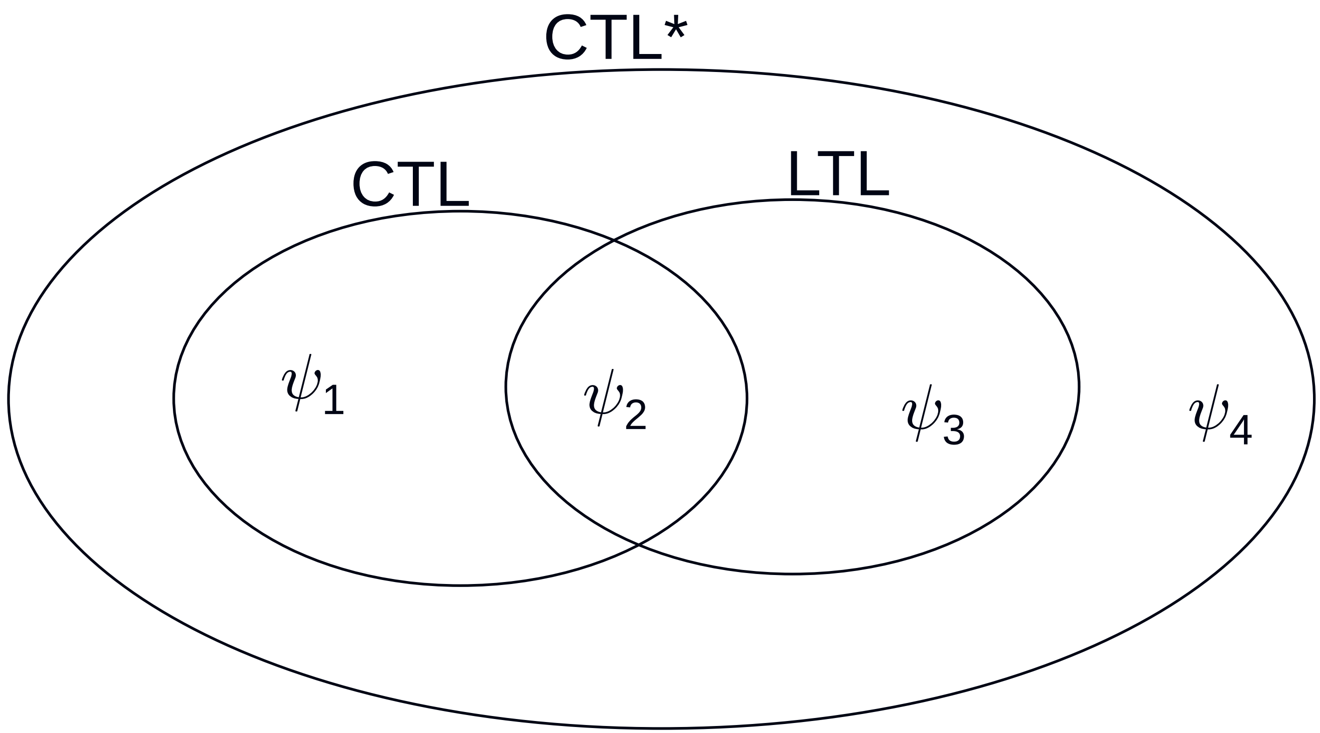




CTL*

- CTL* combines the expressive powers of LTL and CTL
- Syntax of CTL*
  - State formula ϕ ::= T | p | $\neg$ ϕ | $\phi \wedge \phi$ | A [α] | E[α]
  - Path formula α ::= ϕ | $\neg$ α | $\alpha \wedge \alpha$ | α U α | G α | F α | X α
- LTL is a subset of CTL*
  - LTL formula α is equivalent to A[α] in CTL*
- CTL is a subset of CTL*
  - We restrict α ::= ϕ U ϕ | G ϕ | F ϕ | X ϕ
    - No boolean connectives in path formula
      - Not real limitation. See page 6
    - No nesting of the path modalities X,F, and G

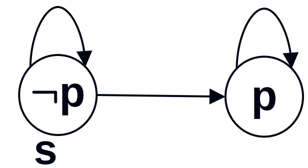
Relationship between LTL,CTL, and CTL*



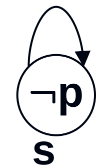
Relationship between LTL,CTL, and CTL*

- In CTL but not in LTL
  - ψ_1 = AG EF p
  - Proof through RAA
    - Let ϕ be an LTL formula s.t. A[ϕ] is equivalent to AG EF p.
    - Since $\mathcal{M}$,s $\vDash$ AG EF p, $\mathcal{M}$,s $\vDash$ A[ϕ]
    - The paths in $\mathcal{M}$' are a subset of those from s in $\mathcal{M}$, so $\mathcal{M}$',s $\vDash$ A[ϕ]
    - However, $\mathcal{M}$',s $\nvDash$ AG EF p. Contradiction

$\mathcal{M}$



$\mathcal{M}$'



- In both CTL and LTL
  - ψ_2 = AG (p $\rightarrow$ AF q) in CTL or G(p $\rightarrow$ F q) in LTL
- In LTL but not in CTL
  - ψ_3 = A [G F p $\rightarrow$ F q]
    - if there are infinitely many p along the path, then there is an occurrence of q
- In CTL* but neither in CTL nor in LTL
  - ψ_4 = E [G F p]
    - there is a path with infinitely many p

Boolean combinations of temporal formulas in CTL

- We can translate CTL* formula having boolean combinations of path formulas (without nested temporal operators) into a CTL formula that does not.
- Examples
  - $E[\ F\ p \wedge F\ q] \equiv EF\ [p \wedge EF\ q] \vee EF\ [q \wedge EF\ p]$
    - If we have $F\ p \wedge F\ q$ along any path, then either the p must come before the q, or the other way around
  - $E\ [\ (p_1\ U\ q_1) \wedge (p_2\ U\ q_2)] \equiv$
    $E[(p_1 \wedge p_2)\ U\ (q_1 \wedge E[p_2\ U\ q_2])] \vee E[(p_1 \wedge p_2) U (q_2 \wedge E[p_1 U q_1])]$
  - $E[\neg(p\ U\ q)] \equiv E[\ \neg q\ U\ (\neg p \wedge \neg q)] \vee EG\ \neg q$
    - since $A\ [p\ U\ q] \equiv \neg(E[\neg q\ U\ (\neg p \wedge \neg q)] \vee EG\ \neg q)$
  - $E[\neg X\ p] \equiv EX\ \neg\ p$

Complexity of Model Checking

- Let $\mathcal{M}$ be a target transition system with *N* states and *M* transitions
- Upper bound of model checking complexity
  - LTL-formula ϕ : $O((N+M)\cdot 2^{|\phi|})$
  - CTL-formula ϕ : $O((N+M)\cdot|\phi|)$
  - CTL*-formula ϕ : $O((N+M)\cdot 2^{|\phi|})$
- Lower bound of model checking complexity
  - LTL-formula ϕ : PSpace-hard -> PSpace-complete
    - Note that P $\subseteq$ NP $\subseteq$ PSpace $\subseteq$ EXP $\subseteq$ EXPSpace
  - CTL-formula ϕ : P-hard -> P-complete
  - CTL*-formula ϕ : PSpace-hard -> PSpace-complete
- For more details, "The Complexity of Temporal Logic Model Checking" by Ph. Schnoebelen
  - *Advances in Modal Logic, Volume 4, 1-44, 2002*