Propositional Calculus
- Natural deduction

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Review

- Goal of logic
  - To check whether given a formula $\phi$ is valid
  - To prove a given formula $\phi$

Sound & Complete

$\vdash \phi$  $\iff$  $\models \phi$

Syntactic method
(proof system $G$, $H$, natural deduction, etc)

Semantic method
(truth table, etc)

Semantic tableau

Sound & Complete
Natural deduction

- A variant of Gentzen system
- In natural deduction, similar to other deductive proof systems such as $\mathcal{G}$ and $\mathcal{H}$, we have a collection of proof rules.
- But natural deduction does not have axioms.
**Proof rules (1/3)**

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Elimination</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \land )</td>
<td>( \land )</td>
</tr>
<tr>
<td>[ \frac{\phi \quad \psi}{\phi \land \psi} ] ( \land i )</td>
<td>[ \frac{\phi \land \psi}{\phi} ] ( \land e_1 )</td>
</tr>
<tr>
<td>[ \frac{\phi \land \psi}{\psi} ] ( \land e_2 )</td>
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</table>

- \( \land i \) says: to prove \( \phi \land \psi \), you must first prove \( \phi \) and \( \psi \) separately and then use the rule \( \land i \).
- \( \land e_1 \) says: to prove \( \phi \), try proving \( \phi \land \psi \) and then use the rule \( \land e_1 \). Actually this does not sound like very good advice because probably proving \( \phi \land \psi \) will be harder than proving \( \phi \) alone. However, you might find that you already have \( \phi \land \psi \) lying around, so that’s when this rule is useful.
Proof rules (1/3)

Introduction

\[
\begin{array}{c}
\phi \\
\phi \lor \psi \\
\phi \lor \psi
\end{array}
\]

\(\lor \text{i}_1\)

\[
\begin{array}{c}
\psi \\
\phi \lor \psi \\
\phi \lor \psi
\end{array}
\]

\(\lor \text{i}_2\)

Elimination

\[
\begin{array}{c}
\phi \lor \psi \\
\chi \\
\chi \\
\phi \lor \psi
\end{array}
\]

\(\lor \text{e}\)

- \(\lor \text{i}_1\) says: to prove \(\phi \lor \psi\), try proving \(\phi\). Again, in general it is harder to prove \(\phi\) than it is to prove \(\phi \lor \psi\), so this will usually be useful only if you have already managed to prove \(\phi\).

- \(\lor \text{e}\) has an excellent procedural interpretation. It says: if you have \(\phi \lor \psi\), and you want to prove some \(\chi\), then try to prove \(\chi\) from \(\phi\) and from \(\psi\) in turn.

- In those subproofs, of course you can use the other prevailing premises as well.
Proof rules (3/3)

**Introduction**

\[ \begin{array}{c}
\phi \\
\vdots \\
\psi \\
\hline
\phi \rightarrow \psi \rightarrow i
\end{array} \]

**Elimination**

\[ \begin{array}{c}
\phi \rightarrow \psi \\
\hline
\psi \rightarrow e
\end{array} \]

\[ \begin{array}{c}
\phi \\
\vdots \\
\bot \\
\hline
\phi \rightarrow l
\end{array} \]

\[ \begin{array}{c}
\bot \\
\hline
\phi \rightarrow c
\end{array} \]

\[ \begin{array}{c}
\phi \\
\vdots \\
\bot \\
\hline
\phi \rightarrow c
\end{array} \]
Some useful derived rules

At any stage of a proof, it is permitted to introduce any formula as assumption, by choosing a proof rule that opens a box. As we saw, natural deduction employs boxes to control the scope of assumptions.

When an assumption is introduced, a box is opened. Discharging assumptions is achieved by closing a box according to the pattern of its particular proof rule.
Example 1

\[ p \land \neg q \rightarrow r, \neg r, p \vdash q \]

1. \[ p \land \neg q \rightarrow r \] premise
2. \[ \neg r \] premise
3. \[ p \] premise
4. \[ \neg q \] assumption
5. \[ p \land \neg q \] \land \text{ i } 3, 4
6. \[ r \] \rightarrow \text{ e } 1, 5
7. \[ \bot \] \neg \text{ e } 6, 2
8. \[ \neg \neg q \] \neg \text{ i } 4–7
9. \[ q \] \neg \neg \text{ e } 8
Example 2

$p \rightarrow q \vdash \neg p \lor q$

1. $p \rightarrow q$ premise
2. $\neg p \lor p$ LEM
3. $\neg p$ assumption
4. $\neg p \lor q \lor \text{I}_1 3$
5. $p$ assumption
6. $q \rightarrow \text{e} 1, 5$
7. $\neg p \lor q \lor \text{I}_2 6$
8. $\neg p \lor q \lor \text{v} \text{e} 2, 3,4, 5,7$
Example 3 (Law of Excluded Middle)

\[ \phi \lor \neg \phi \quad \text{LEM} \]

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<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>(\neg (\phi \lor \neg \phi))</td>
<td>assumption</td>
</tr>
<tr>
<td>2</td>
<td>(\phi)</td>
<td>assumption</td>
</tr>
<tr>
<td>3</td>
<td>(\phi \lor \neg \phi)</td>
<td>(\lor_i 1) 2</td>
</tr>
<tr>
<td>4</td>
<td>(\bot)</td>
<td>(\neg e) 3, 1</td>
</tr>
<tr>
<td>5</td>
<td>(\neg \phi)</td>
<td>(\neg i) 2–4</td>
</tr>
<tr>
<td>6</td>
<td>(\phi \lor \neg \phi)</td>
<td>(\lor_i 2) 5</td>
</tr>
<tr>
<td>7</td>
<td>(\bot)</td>
<td>(\neg e) 6, 1</td>
</tr>
<tr>
<td>8</td>
<td>(\neg (\phi \lor \neg \phi))</td>
<td>(\neg i) 1–7</td>
</tr>
<tr>
<td>9</td>
<td>(\phi \lor \neg \phi)</td>
<td>(\neg e) 8</td>
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Summary of proof rules

<table>
<thead>
<tr>
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<tr>
<td>$\land$</td>
<td></td>
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<tr>
<td>$\phi \land \psi$ \quad $\land_i$</td>
<td>$\phi \land \psi$ \quad $\land_e_1$</td>
</tr>
<tr>
<td>$\phi \lor \psi$ \quad $\lor_i_1$</td>
<td>$\phi \lor \psi$ \quad $\lor_i_2$</td>
</tr>
<tr>
<td>$\phi \quad \psi$</td>
<td>$\phi \quad \psi$</td>
</tr>
<tr>
<td>$\vdash$</td>
<td></td>
</tr>
<tr>
<td>$\phi \rightarrow \psi$ \quad $\rightarrow_i$</td>
<td>$\phi \phi \rightarrow \psi$ \quad $\rightarrow_e$</td>
</tr>
<tr>
<td>$\phi$ \quad $\vdash$</td>
<td>$\phi$ \quad $\vdash$</td>
</tr>
<tr>
<td>$\neg \phi$ \quad $\neg_i$</td>
<td>$\phi \neg \phi$ \quad $\neg_e$</td>
</tr>
<tr>
<td>$\bot$ \quad (no introduction rule for $\bot$)</td>
<td>$\bot$ \quad $\bot_e$</td>
</tr>
<tr>
<td>$\bot$ \quad $\bot$</td>
<td>$\bot$ \quad $\bot$</td>
</tr>
<tr>
<td>$\neg \phi$ \quad $\neg_i$</td>
<td>$\neg \phi$ \quad $\neg_i$</td>
</tr>
<tr>
<td>$\phi$ \quad $\neg \phi$ \quad $\neg_e$</td>
<td></td>
</tr>
<tr>
<td>$\neg \phi$ \quad $\neg_e$</td>
<td>$\phi \lor \neg \phi$ \quad $\lor_i$</td>
</tr>
<tr>
<td>$\bot$ \quad $\bot_e$</td>
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