Propositional Calculus
- *Soundness & Completeness of $\mathcal{H}$*

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Review

- Goal of logic
  - To check whether given a formula ϕ is valid
  - To prove a given formula ϕ

Sound & Complete

$\vdash \phi$ $\equiv$ $\models \phi$

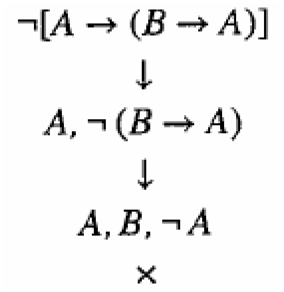
Syntactic method (proof system $\mathcal{G}$, $\mathcal{H}$, natural deduction,etc)

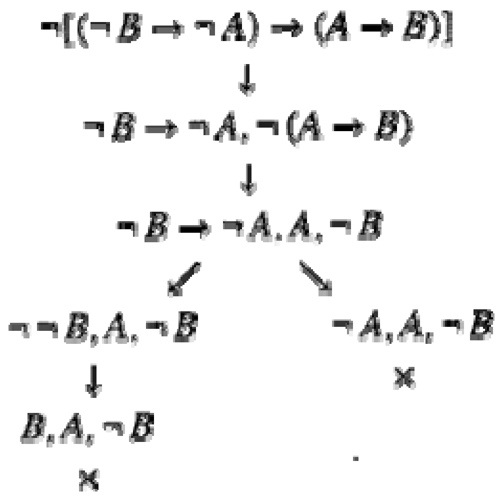
Semantic method (truth table, etc)

Sound & Complete $\equiv$ **Semantic tableau** $\equiv$ **Sound & Complete**

Soundness of $\mathcal{H}$ (1/2)

- Thm 3.34 $\mathcal{H}$ is sound, that is $\vdash$ A then $\models$ A
  - Proof is by structural induction
  - We show that
    1. the all three axioms are valid and that
    2. if the premises of MP are valid, so is the conclusion
- Task 1: to prove $\models$ Axiom1, $\models$ Axiom2, and $\models$ Axiom3
  - By showing the semantic tableau of the negated axiom is closed





Soundness of $\mathcal{H}$ (2/2)

- Task 2: proof by RAA (귀류법)
  - Suppose that MP were not sound.
    - Then there would be a set of formulas {A, A $\rightarrow$ B, B} such that A and A $\rightarrow$ B are valid, but B is not valid

$$\frac{\vdash A \qquad \vdash A \rightarrow B}{\vdash B}$$

    - If B is not valid, there is an interpretation v such that v(B) = F. Since A and A $\rightarrow$ B are valid, for any interpretation, in particular for v, v(A) = v(A $\rightarrow$ B) = T. From this we deduce that v(B) = T contradicting the choice of v

Completeness of $\mathcal{H}$(¼)

- Thm 3.35 $\mathcal{H}$ is complete, that is, if $\models A$ then $\vdash A$
- Any valid formula can be proved in $\mathcal{G}$ (thm 3.8). We will show how a proof in $\mathcal{G}$ can be mechanically transformed into a proof in $\mathcal{H}$
- The exact correspondence is that if the set of formulas U is provable in $\mathcal{G}$ then the single formula $\vee$U is provable in $\mathcal{H}$
  - A problem is that
    - We can show that $\{\neg p, p\}$ is an axiom in $\mathcal{G}$ then $\vdash p \vee \neg p$ in $\mathcal{H}$ since this is simply Thm 3.10 ($\vdash A \rightarrow A$)
      - Note that $A \vee B$ is an abbreviation for $\neg A \rightarrow B$
      - Similarly $A \wedge B$ is an abbreviation for $\neg (A \rightarrow \neg B)$
    - But if the axiom in $\mathcal{G}$ is $\{q, \neg p, r, p, s\}$, we cannot immediately conclude that $\vdash q \vee \neg p \vee r \vee p \vee s$

Completeness of $\mathcal{H}$(2/4)

- Lem 3.36 If U'⊆U and ⊢ ∨U' (in $\mathcal{H}$) then ⊢ ∨U (in $\mathcal{H}$)
- The proof is by induction using Thm 3.31 through 3.33
  - Suppose we have a proof of ∨ U'. By repeated application of Thm 3.31, we can transform this into a proof of ∨ U", where U" is a permutation of the elements of U.
    - Thm 3.31 Weakening ⊢ A → A ∨ B and ⊢ B → A ∨ B
  - Now by repeated applications of the commutativity and associativity of disjunction, we can move the elements of U" to their proper places
    - Thm 3.32 Commutativity rule: ⊢ A ∨ B ↔ B ∨ A
    - Thm 3.33 Associativity rule : ⊢ A ∨ (B ∨ C) ↔ (A ∨ B) ∨ C

Completeness of $\mathcal{H}$(3/4)

- Completeness proof by induction on the structure of the proof in $\mathcal{G}$
  - We are transforming a proof in $\mathcal{G}$ to a proof in $\mathcal{H}$
- Task 1:
  - If U is an axiom, it contains a pair of complementary literals and $\vdash \neg p \vee p$ can be proved in $\mathcal{H}$. BY Lem 3.36, this may be transformed into a proof of $\vee$ U.
  - Lem 3.36 If $U' \subseteq U$ and $\vdash \vee U'$ (in $\mathcal{H}$) then $\vdash \vee U$ (in $\mathcal{H}$)

Completeness of $\mathcal{H}$(4/4)

- Task 2:
  - The last step in the proof of U in $\mathcal{G}$ is the application of an α or β rule.
    - Case 1: An α rule was used in $\mathcal{G}$ to infer $\dfrac{\vdash U_1 \cup \{A_1, A_2\}}{\vdash U_1 \cup \{A_1 \vee A_2\}}$
      - By the inductive hypothesis, $\vdash (\vee U_1 \vee A_1) \vee A_2$ in $\mathcal{H}$ from which we infer $\vdash \vee U_1 \vee (A_1 \vee A_2)$ by associativity
    - Case 2: An β rule was used in $\mathcal{G}$ to infer

$$\frac{\vdash U_1 \cup \{A_1\} \quad \vdash U_2 \cup \{A_2\}}{\vdash U_1 \cup U_2 \cup \{A_1 \wedge A_2\}}$$

      - By the inductive hypothesis, $\vdash \vee U_1 \vee A_1$ and $\vdash \vee U_2 \vee A_2$ in $\mathcal{H}$. Thus $\vdash \vee U_1 \vee \vee U_2 \vee (A_1 \wedge A_2)$

Consistency

- Def 3.38 A set of formulas U is inconsistent iff for some formula A, $U \vdash A$ and $U \vdash \neg A$. U is consistent iff it is not inconsistent
- Thm 3.39 U is inconsistent iff for all A, $U \vdash A$
  - Proof: Let A be an arbitrary formula. Since U is incosistent, for some formula B, $U \vdash B$ and $U \vdash \neg B$.
  - By Thm 3.21 $\vdash B \rightarrow (\neg B \rightarrow A)$. Using MP twice, $U \vdash A$.
- Corollary 3.40 U is consistent iff for some A, $U \nvdash A$
- Them 3.41 $U \vdash A$ iff $U \cup \{\neg A\}$ is inconsistent