

# Predicate Calculus

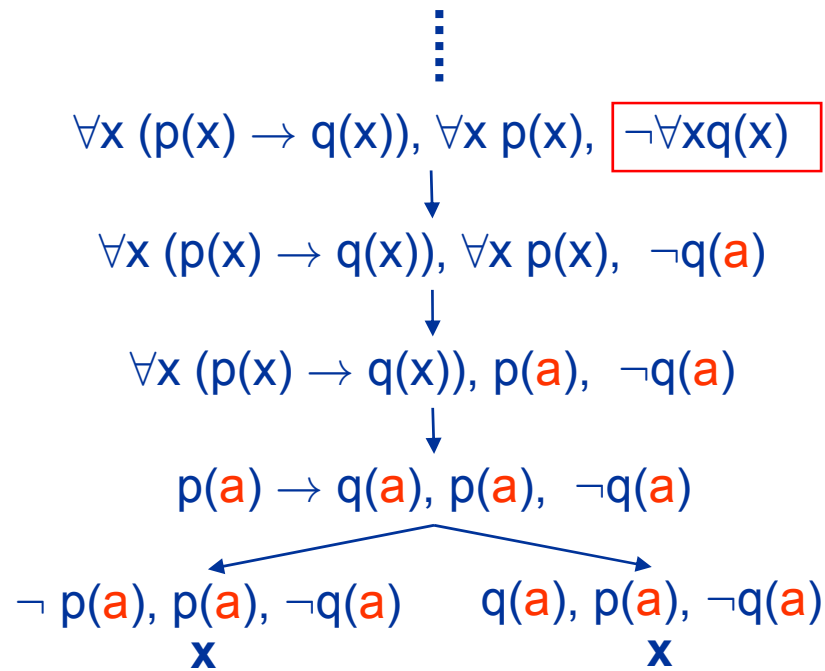
## - Semantic Tableau (1/2)

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## Informal construction of a valid formula (2/2)

- Note that semantic tableau is to find **a single counter example**
  - $\neg \forall x q(x) \equiv \exists x \neg q(x)$
  - Therefore, we could replace a variable  $x$  in  $\neg \forall x q(x)$  by a single concrete element **a** in the target domain
    - In other words, we use  $\neg q(a)$  instead of  $\neg \forall x q(x)$



# Informal construction of a satisfiable formula (1/3)

- Example 2: a satisfiable but not valid formula

- $\forall x (p(x) \vee q(x)) \rightarrow (\forall x p(x) \vee \forall x q(x))$

$$\neg (\forall x (p(x) \vee q(x)) \rightarrow (\forall x p(x) \vee \forall x q(x)))$$



$$\forall x (p(x) \vee q(x)), \neg (\forall x p(x) \vee \forall x q(x))$$



$$\forall x (p(x) \vee q(x)), \neg \forall x p(x), \neg \forall x q(x)$$



$$\forall x (p(x) \vee q(x)), \neg \forall x p(x), \neg q(a)$$



$$\forall x (p(x) \vee q(x)), \neg p(a), \neg q(a)$$



$$p(a) \vee q(a), \neg p(a), \neg q(a)$$



$$p(a), \neg p(a), \neg q(a)$$

**x**

$$q(a), \neg p(a), \neg q(a)$$

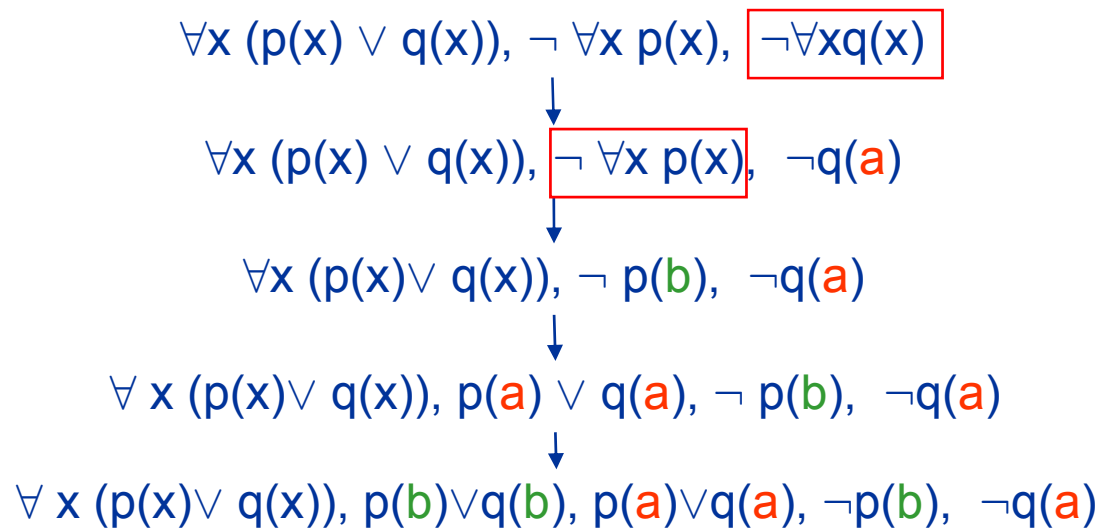
**x**

$\alpha$	$\alpha_1$	$\alpha_2$
$\neg \neg A_1$	$A_1$	
$A_1 \wedge A_2$	$A_1$	$A_2$
$\neg (A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg (A_1 \rightarrow A_2)$	$A_1$	$\neg A_2$
$\neg (A_1 \uparrow A_2)$	$A_1$	$A_2$
$A_1 \downarrow A_2$	$\neg A_1$	$\neg A_2$
$A_1 \leftrightarrow A_2$	$A_1 \rightarrow A_2$	$A_2 \rightarrow A_1$
$\neg (A_1 \oplus A_2)$	$A_1 \rightarrow A_2$	$A_2 \rightarrow A_1$

$\beta$	$\beta_1$	$\beta_2$
$\neg (B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \vee B_2$	$B_1$	$B_2$
$B_1 \rightarrow B_2$	$\neg B_1$	$B_2$
$B_1 \uparrow B_2$	$\neg B_1$	$\neg B_2$
$\neg (B_1 \downarrow B_2)$	$B_1$	$B_2$
$\neg (B_1 \leftrightarrow B_2)$	$\neg (B_1 \rightarrow B_2)$	$\neg (B_2 \rightarrow B_1)$
$B_1 \oplus B_2$	$\neg (B_1 \rightarrow B_2)$	$\neg (B_2 \rightarrow B_1)$

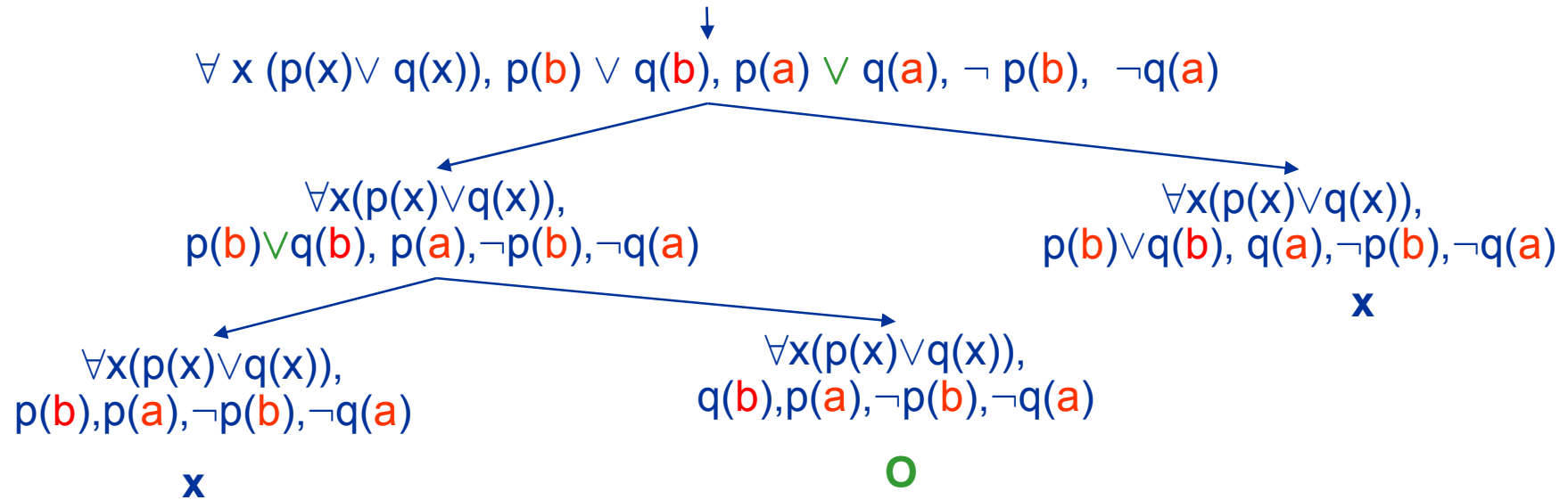
# Informal construction of a satisfiable formula (2/3)

- What is wrong?
  - 1. Use different constants for different formulas
    - It is ok to use  $\neg q(a)$  instead of  $\neg \forall x q(x)$
    - However, it is **not** ok to use the **same** element  $a$  for a different formula  $\neg \forall x p(x)$
  - 2. A formula with universal quantifiers without negation **cannot** be simply replaced by just one instance
    - Universal formulas should never be deleted from the node.
    - Universal formulas remain in the all descendant nodes so as to constrain the possible interpretations of **every new constant** that is introduced.



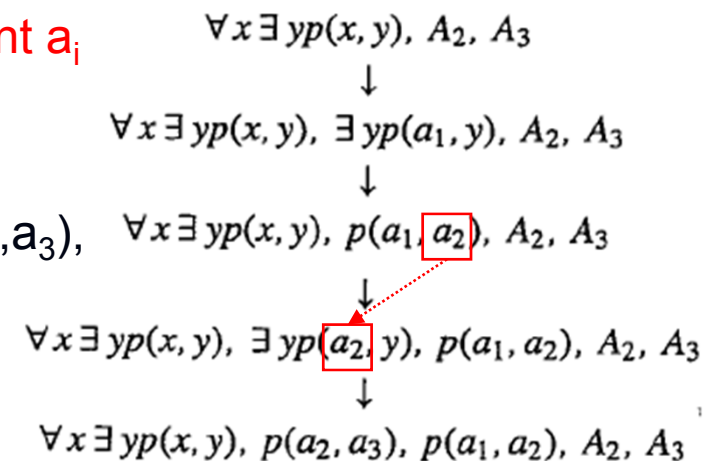
# Informal construction of a satisfiable formula (3/3)

- The following formula is satisfiable but not valid
  - $\forall x (p(x) \vee q(x)) \rightarrow (\forall x p(x) \vee \forall x q(x))$



# Infinite construction (1/3)

- $A = A_1 \wedge A_2 \wedge A_3$ 
  - $A_1 = \forall x \exists y p(x,y)$
  - $A_2 = \forall x \neg p(x,x)$
  - $A_3 = \forall xyz (p(x,y) \wedge p(y,z) \rightarrow p(x,z))$
- Note that we do **not** have a **constant** in  $A$
- The construction will **not** terminate
  - If we continue the tableau construction, an infinite branch is obtained
  - The tableau neither closes nor terminates
  - It defines an countably infinite model
    - Note that once we introduce **a new constant  $a_i$**  by instantiating  $\exists y$ , then  $\forall x$  **should** be instantiated with **that constant  $a_i$**
    - Therefore, semantic tableau will have an **infinite sequence** of formulas  $p(a_1,a_2), p(a_2,a_3), p(a_3,a_4), \dots$



# Infinite construction (2/3)

- Thm 5.24.  $A = A_1 \wedge A_2 \wedge A_3$  has no finite model
  - $A_1 = \forall x \exists y p(x,y)$
  - $A_2 = \forall x \neg p(x,x)$
  - $A_3 = \forall xyz (p(x,y) \wedge p(y,z) \rightarrow p(x,z))$
  - Suppose that  $A$  had a **finite** model
    - The domain of an interpretation is non-empty so it has at least one element.
    - By  $A_1$ , there is an **infinite** sequence of elements  $a_1, a_2, \dots$  s.t.  $\forall_{\sigma_{\mathcal{I}}[x \leftarrow a_i][y \leftarrow a_j]} (p(x,y)) = T$  for all  $i$  and  $j=i+1$ .
    - By  $A_3$ ,  $p(a_i, a_j) = T$  for all  $j > i$  since  $A_3$  means transitivity
      - i.e.,  $p(a_1, a_2) \wedge p(a_2, a_3) \rightarrow p(a_1, a_3)$
    - Since we assume that the model is finite, **there exists some  $k > i$**  such that  **$a_k = a_i$**  due to pigeon hole principle.
      - Note that we have an infinite sequence of elements by  $A_1$ . But the model has only finite elements.
    - For some  $k > i$  s.t.  $a_k = a_i$ ,  $p(a_i, a_k) = T$  by  $A_3$ . This **contradicts**  $A_2$  which requires  $\forall_{\sigma_{\mathcal{I}}[x \leftarrow a_i]} (p(x,x)) = F$ .



# Infinite construction (3/3)

- Note that construction of semantic tableaux is **not** a decision procedure for validity in the predicate calculus as we have seen the previous example.
- Also, note that without **systematic construction**, we may **not** construct a closed semantic tableaux even when it is possible.
  - In the following example, if we choose the last formula, we can close the tableau immediately. If we choose  $A_1$ , however, we will have an infinite branch.

$$\begin{array}{c} A_1 \wedge A_2 \wedge A_3 \wedge \forall x (q(x) \wedge \neg q(x)) \\ \downarrow \\ A_1, A_2, A_3, \forall x (q(x) \wedge \neg q(x)) \end{array}$$