Predicate Calculus
- Semantic Tableau (1/2)

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Informal construction of a valid formula (1/2)

- Example 1: a valid formula
  - $\forall x\,(p(x) \to q(x)) \to (\forall x\, p(x) \to \forall x q(x))$

$$\neg(\forall x\,(p(x) \to q(x)) \to (\forall x\, p(x) \to \forall x\, q(x)))$$

$$\downarrow$$

$$\forall x\,(p(x) \to q(x)),\ \neg(\forall x\, p(x) \to \forall x\, q(x)))$$

$$\downarrow$$

$$\forall x\,(p(x) \to q(x)),\ \forall x\, p(x),\ \boxed{\neg\forall x q(x)}$$

$$\downarrow$$

$$\forall x\,(p(x) \to q(x)),\ \forall x\, p(x),\ \neg q(a)$$

$$\downarrow$$

$$\forall x\,(p(x) \to q(x)),\ p(a),\ \neg q(a)$$

$$\downarrow$$

$$p(a) \to q(a),\ p(a),\ \neg q(a)$$

$$\swarrow \qquad \searrow$$

$$\neg p(a),\ p(a),\ \neg q(a) \qquad\qquad q(a),\ p(a),\ \neg q(a)$$

$$\times \qquad\qquad\qquad\qquad \times$$

α	α_1	α_2
$\neg\neg A_1$	A_1	
$A_1 \wedge A_2$	A_1	A_2
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \to A_2)$	A_1	$\neg A_2$
$\neg(A_1 \uparrow A_2)$	A_1	A_2
$A_1 \downarrow A_2$	$\neg A_1$	$\neg A_2$
$A_1 \leftrightarrow A_2$	$A_1 \to A_2$	$A_2 \to A_1$
$\neg(A_1 \oplus A_2)$	$A_1 \to A_2$	$A_2 \to A_1$

β	β_1	β_2
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \vee B_2$	B_1	B_2
$B_1 \to B_2$	$\neg B_1$	B_2
$B_1 \uparrow B_2$	$\neg B_1$	$\neg B_2$
$\neg(B_1 \downarrow B_2)$	B_1	B_2
$\neg(B_1 \leftrightarrow B_2)$	$\neg(B_1 \to B_2)$	$\neg(B_2 \to B_1)$
$B_1 \oplus B_2$	$\neg(B_1 \to B_2)$	$\neg(B_2 \to B_1)$

Informal construction of a valid formula (2/2)

- Note that semantic tableau is to find a single counter example
  - $\neg \forall x\, q(x) \equiv \exists x\, \neg q(x)$
  - Therefore, we could replace a variable x in $\neg \forall x\, q(x)$ by a single concrete element a in the target domain
    - In other words, we use $\neg q(a)$ instead of $\neg \forall x\, q(x)$

$$\vdots$$

$$\forall x\,(p(x) \rightarrow q(x)),\ \forall x\, p(x),\ \boxed{\neg\forall x q(x)}$$

$$\downarrow$$

$$\forall x\,(p(x) \rightarrow q(x)),\ \forall x\, p(x),\ \neg q(a)$$

$$\downarrow$$

$$\forall x\,(p(x) \rightarrow q(x)),\ p(a),\ \neg q(a)$$

$$\downarrow$$

$$p(a) \rightarrow q(a),\ p(a),\ \neg q(a)$$

$$\swarrow \qquad \searrow$$

$$\underset{\times}{\neg p(a),\ p(a),\ \neg q(a)} \qquad \underset{\times}{q(a),\ p(a),\ \neg q(a)}$$

Informal construction of a satisfiable formula (1/3)

- Example 2: a satisfiable but not valid formula
  - $\forall x\,(p(x) \vee q(x)) \rightarrow (\forall x\, p(x) \vee \forall x\, q(x))$

$$\neg\,(\forall x\,(p(x) \vee q(x)) \rightarrow (\forall x\, p(x) \vee \forall x\, q(x)))$$

$$\downarrow$$

$$\forall x\,(p(x) \vee q(x)),\ \neg\,(\forall x\, p(x) \vee \forall x\, q(x)))$$

$$\downarrow$$

$$\forall x\,(p(x) \vee q(x)),\ \neg\,\forall x\, p(x),\ \boxed{\neg\forall x q(x)}$$

$$\downarrow$$

$$\forall x\,(p(x) \vee q(x)),\ \neg\,\forall x\, p(x),\ \neg q(a)$$

$$\downarrow$$

$$\forall x\,(p(x) \vee q(x)),\ \neg\, p(a),\ \neg q(a)$$

$$\downarrow$$

$$p(a) \vee q(a),\ \neg\, p(a),\ \neg q(a)$$

$$\swarrow \qquad \searrow$$

$$p(a),\ \neg p(a),\ \neg q(a) \qquad q(a),\ \neg\, p(a),\ \neg q(a)$$

$$\times \qquad\qquad \times$$

α	α_1	α_2
$\neg\neg A_1$	A_1	
$A_1 \wedge A_2$	A_1	A_2
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \rightarrow A_2)$	A_1	$\neg A_2$
$\neg(A_1 \uparrow A_2)$	A_1	A_2
$A_1 \downarrow A_2$	$\neg A_1$	$\neg A_2$
$A_1 \leftrightarrow A_2$	$A_1 \rightarrow A_2$	$A_2 \rightarrow A_1$
$\neg(A_1 \oplus A_2)$	$A_1 \rightarrow A_2$	$A_2 \rightarrow A_1$

β	β_1	β_2
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \vee B_2$	B_1	B_2
$B_1 \rightarrow B_2$	$\neg B_1$	B_2
$B_1 \uparrow B_2$	$\neg B_1$	$\neg B_2$
$\neg(B_1 \downarrow B_2)$	B_1	B_2
$\neg(B_1 \leftrightarrow B_2)$	$\neg(B_1 \rightarrow B_2)$	$\neg(B_2 \rightarrow B_1)$
$B_1 \oplus B_2$	$\neg(B_1 \rightarrow B_2)$	$\neg(B_2 \rightarrow B_1)$

Informal construction of a satisfiable formula (2/3)

- What is wrong?
  - 1. Use different constants for different formulas
    - It is ok to use $\neg q(a)$ instead of $\neg\forall x\, q(x)$
    - However, it is not ok to use the same element a for a different formula $\neg\forall x\, p(x)$
  - 2.A formula with universal quantifiers without negation cannot be simply replaced by just one instance
    - Universal formulas should never be deleted from the node.
    - Universal formulas remain in the all descendant nodes so as to constrain the possible interpretations of every new constant that is introduced.

$$\forall x\,(p(x) \vee q(x)),\ \neg \forall x\, p(x),\ \boxed{\neg\forall x q(x)}$$

$$\downarrow$$

$$\forall x\,(p(x) \vee q(x)),\ \boxed{\neg \forall x\, p(x)},\ \neg q(a)$$

$$\downarrow$$

$$\forall x\,(p(x)\vee q(x)),\ \neg p(b),\ \neg q(a)$$

$$\downarrow$$

$$\forall x\,(p(x)\vee q(x)),\ p(a) \vee q(a),\ \neg p(b),\ \neg q(a)$$

$$\downarrow$$

$$\forall x\,(p(x)\vee q(x)),\ p(b)\vee q(b),\ p(a)\vee q(a),\ \neg p(b),\ \neg q(a)$$

Informal construction of a satisfiable formula (3/3)

- The following formula is satisfiable but not valid
  - $\forall x\,(p(x) \lor q(x)) \rightarrow (\forall x\, p(x) \lor \forall x q(x))$

$\downarrow$

$\forall x\,(p(x) \lor q(x)),\ p(b) \lor q(b),\ p(a) \lor q(a),\ \neg p(b),\ \neg q(a)$

$\forall x(p(x) \lor q(x)),$
$p(b) \lor q(b),\ p(a), \neg p(b), \neg q(a)$

$\forall x(p(x) \lor q(x)),$
$p(b) \lor q(b),\ q(a), \neg p(b), \neg q(a)$

x

$\forall x(p(x) \lor q(x)),$
$p(b), p(a), \neg p(b), \neg q(a)$

x

$\forall x(p(x) \lor q(x)),$
$q(b), p(a), \neg p(b), \neg q(a)$

O

Infinite construction (1/3)

- $A = A_1 \wedge A_2 \wedge A_3$
  - $A_1 = \forall x\, \exists y\, p(x,y)$
  - $A_2 = \forall x\, \neg p(x,x)$
  - $A_3 = \forall xyz\, (p(x,y) \wedge p(y,z) \rightarrow p(x,z))$
- Note that we do not have a constant in A
- The construction will not terminate
  - If we continue the tableau construction, an infinite branch is obtained
  - The tableau neither closes nor terminates
  - It defines an countably infinite model
    - Note that once we introduce a new constant a_i by instantiating $\exists y$, then $\forall x$ should be instantiated with that constant a_i
    - Therefore, semantic tableau will have an infinite sequence of formulas $p(a_1,a_2)$, $p(a_2,a_3)$, $p(a_3,a_4)$, …

$$\forall x \exists y p(x,y),\ A_2,\ A_3$$
$$\downarrow$$
$$\forall x \exists y p(x,y),\ \exists y p(a_1,y),\ A_2,\ A_3$$
$$\downarrow$$
$$\forall x \exists y p(x,y),\ p(a_1,\boxed{a_2}),\ A_2,\ A_3$$
$$\downarrow$$
$$\forall x \exists y p(x,y),\ \exists y p(\boxed{a_2},y),\ p(a_1,a_2),\ A_2,\ A_3$$
$$\downarrow$$
$$\forall x \exists y p(x,y),\ p(a_2,a_3),\ p(a_1,a_2),\ A_2,\ A_3$$

Infinite construction (2/3)

- Thm 5.24. $A = A_1 \wedge A_2 \wedge A_3$ has no finite model
  - $A_1 = \forall x \exists y\, p(x,y)$
  - $A_2 = \forall x\, \neg p(x,x)$
  - $A_3 = \forall xyz\, (p(x,y) \wedge p(y,z) \rightarrow p(x,z))$
  - Suppose that A had a finite model
    - The domain of an interpretation is non-empty so it has at leas one element.
    - By A_1, there is an infinite sequence of elements $a_1, a_2, \ldots$ s.t. $v_{\sigma_{\mathcal{I}}[x \leftarrow a_i][y \leftarrow a_j]}(p(x,y)) = T$ for all i and j=i+1.
    - By A_3, $p(a_i, a_j) = T$ for all $j > i$ since A_3 means transitivity
      - i.e., $p(a_1,a_2) \wedge p(a_2,a_3) \rightarrow p(a_1, a_3)$
    - Since we assume that the model is finite, there exists some $k > i$ such that $a_k = a_i$ due to pigeon hole principle.
      - Note that we have an infinite sequence of elements by A_1. But the model has only finite elements.
    - For some $k > i$ s.t. $a_k = a_i$, $p(a_i, a_k) = T$ by A_3. This contradicts A_2 which requires $v_{\sigma_{\mathcal{I}}[x \leftarrow a_i]}(p(x,x)) = F$.

Infinite construction (3/3)

- Note that construction of semantic tableaux is not a decision procedure for validity in the predicate calculus as we have seen the previous example.
- Also, note that without systematic construction, we may not construct a closed semantic tableaux even when it is possible.
  - In the following example, if we choose the last formula, we can close the tableau immediately. If we choose A_1, however, we will have an infinite branch.

$$A_1 \wedge A_2 \wedge A_3 \wedge \forall x\, (q(x) \wedge \neg q(x))$$

$$\downarrow$$

$$A_1, A_2, A_3, \forall x (q(x) \wedge \neg q(x))$$