

# Predicate Calculus - Natural deduction

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# Natural deduction

- Proofs in the natural deduction for predicate logic are similar to those for propositional logic
  - We have new proof rules for dealing with  $\forall, \exists$  and with the **equality (=)** symbol
  - As in the natural deduction for propositional logic, the additional rules for the quantifiers and equality will come in two flavors
    - **introduction** and **elimination** rules

# Summary of proof rules of natural deduction

	introduction	elimination		
$\wedge$	$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\phi \wedge \psi}{\psi} \wedge e_2$	$\boxed{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}}$	<b>assumption</b>
$\vee$	$\frac{\phi}{\phi \vee \psi} \vee i_1 \quad \frac{\psi}{\phi \vee \psi} \vee i_2$	$\frac{\phi \vee \psi \quad \boxed{\begin{array}{c} \phi \\ \vdots \\ \chi \end{array}} \quad \boxed{\begin{array}{c} \psi \\ \vdots \\ \chi \end{array}}}{\chi} \vee e$		
$\rightarrow$	$\boxed{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}} \rightarrow i$	$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e$	$\boxed{\begin{array}{c} \neg \phi \\ \vdots \\ \perp \end{array}}$	<b>assumption</b>
$\neg$	$\boxed{\begin{array}{c} \phi \\ \vdots \\ \perp \end{array}} \neg i$	$\frac{\phi \quad \neg \phi}{\perp} \neg e$	$\frac{\phi \rightarrow \psi \quad \neg \psi}{\neg \phi} \text{MT}$	$\frac{\phi}{\neg \neg \phi} \neg \neg i$
$\perp$	(no introduction rule for $\perp$ )	$\frac{\perp}{\phi} \perp e$	$\frac{\perp}{\phi} \text{RAA}$	$\frac{}{\phi \vee \neg \phi} \text{LEM}$
$\neg \neg$		$\frac{\neg \neg \phi}{\phi} \neg \neg e$		

# Example 1

$p \wedge \neg q \rightarrow r, \neg r, p \vdash q$

1	$p \wedge \neg q \rightarrow r$	premise
2	$\neg r$	premise
3	$p$	premise
4	$\neg q$	assumption
5	$p \wedge \neg q$	$\wedge$ i 3, 4
6	$r$	$\rightarrow$ e 1, 5
7	$\perp$	$\neg$ e 6, 2
8	$\neg\neg q$	$\neg$ i 4–7
9	$q$	$\neg\neg$ e 8

## Example 2

$$p \rightarrow q \vdash \neg p \vee q$$

1	$p \rightarrow q$	premise
2	$\neg p \vee p$	LEM
3	$\neg p$	assumption
4	$\neg p \vee q$	$\vee i_1$ 3
5	$p$	assumption
6	$q$	$\rightarrow e$ 1, 5
7	$\neg p \vee q$	$\vee i_2$ 6
8	$\neg p \vee q$	$\vee e$ 2, 3–4, 5–7

# Example 3 (Law of Excluded Middle)

$\overline{\phi \vee \neg\phi}$  LEM

1	$\neg(\phi \vee \neg\phi)$	assumption
2	$\phi$	assumption
3	$\phi \vee \neg\phi$	$\vee i_1$ 2
4	$\perp$	$\neg e$ 3, 1
5	$\neg\phi$	$\neg i$ 2–4
6	$\phi \vee \neg\phi$	$\vee i_2$ 5
7	$\perp$	$\neg e$ 6, 1
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg i$ 1–7
9	$\phi \vee \neg\phi$	$\neg\neg e$ 8

# The proof rules for $\forall$ and $\exists$

$$\frac{\boxed{\begin{array}{c} x_0 \\ \vdots \\ \phi[x_0/x] \end{array}}}{\forall x \phi} \forall x i$$

$$\frac{\forall x \phi}{\phi[t/x]} \forall x e$$

$$\frac{\phi[t/x]}{\exists x \phi} \exists x i$$

$$\frac{\boxed{\begin{array}{cc} x_0 & \phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi} \exists x e$$

# $\forall x e, \forall x i$

- $\forall x e$ 
  - If  $\forall x \phi$  is true, then you could replace the  $x$  in  $\phi$  by any term  $t$ 
    - $t$  must be **free** for  $x$  in  $\phi$
  - Ex. Let  $\phi = \exists y (x < y)$ 
    - Suppose that we replace  $x$  with  $y$ , i.e.,  $\phi[y/x] = \exists y (y < y)$
    - very different meaning!

$$\frac{\forall x \phi}{\phi[t/x]} \forall x e$$

- $\forall x i$ 
  - If, starting with a **fresh** variable  $x_0$ , you are able to prove some formula  $\phi[x_0/x]$  with  $x_0$  in it, then (because  $x_0$  is fresh) you can derive  $\forall x \phi$
  - $x_0$  does **not** occur outside the box

$$\frac{\boxed{\begin{array}{c} x_0 \\ \vdots \\ \phi[x_0/x] \end{array}}}{\forall x \phi} \forall x i$$



# Example

■  $\forall x (P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x)$

①  $\forall x (P(x) \rightarrow Q(x))$  Premise

②  $\forall x P(x)$  Premise

③  $x_0$   $P(x_0) \rightarrow Q(x_0)$   $\forall x e 1$

④  $P(x_0)$   $\forall x e 2$

⑤  $Q(x_0)$   $\rightarrow e 3,4$

⑥  $\forall x Q(x)$   $\forall x i 3-5$

# $\exists x i, \exists x e$

- $\exists x i$ 
  - It simply says that we can deduce  $\exists x \phi$  whenever we have  $\phi[t/x]$  for some term  $t$ 
    - $t$  must be free for  $x$  in  $\phi$
- $\exists x e$ 
  - We know  $\exists x \phi$  is true, so  $\phi$  is true for at least one value of  $x$ 
    - So we do a case analysis over all those possible values, writing  $x_0$  as a generic value representing them all

$$\frac{\phi [t / x]}{\exists x \phi} \exists x i$$

$$\frac{\exists x \phi \quad \boxed{\begin{array}{c} x_0 \quad \phi[x_0/x] \\ \vdots \\ \chi \end{array}}}{\chi} \exists x e$$

# Example

- $\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$
- ①  $\forall x (P(x) \rightarrow Q(x))$  Premise
- ②  $\exists x P(x)$  Premise
- ③  $x_0 P(x_0)$  Assumption
- ④  $P(x_0) \rightarrow Q(x_0)$   $\forall x e 1$
- ⑤  $Q(x_0)$   $\rightarrow e 4,3$
- ⑥  $\exists x Q(x)$   $\exists x i 5$
- ⑦  $\exists x Q(x)$   $\exists x e 2,3-6$