Predicate Calculus
- Natural deduction (2/2)

Moonzoo Kim
CS Division of EECS Dept.
KAIST

Summary of proof rules of natural deduction

	introduction	elimination
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$ (assumption)
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$ (assumption)	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$ (assumption)	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

$$\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad \dfrac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad \dfrac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

assumption

$$\dfrac{\boxed{\begin{array}{l}x_0\\ \quad\vdots\\ \quad\phi[x_0/x]\end{array}}}{\forall x\, \phi}\ \forall x\ i \qquad \dfrac{\forall x\, \phi}{\phi[t/x]}\ \forall x\ e$$

$$\dfrac{\phi[t/x]}{\exists x\, \phi}\ \exists x\ i \qquad \dfrac{\exists x\, \phi \quad \boxed{\begin{array}{l}x_0 \quad \phi[x_0/x]\\ \qquad \vdots\\ \qquad \chi\end{array}}}{\chi}\ \exists x e$$

assumption

Example

$\forall x\,(P(x) \rightarrow Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$

①		$\forall x\,(P(x) \rightarrow Q(x))$	Premise
②		$\exists x\,P(x)$	Premise
③	x_0	$P(x_0)$	Assumption
④		$P(x_0) \rightarrow Q(x_0)$	$\forall$x e 1
⑤		$Q(x_0)$	$\rightarrow$e 4,3
⑥		$\exists x\,Q(x)$	$\exists$x i 5
⑦		$\exists x\,Q(x)$	$\exists$x e 2,3-6

$\forall x\,(P(x) \rightarrow Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$

①		$\forall x\,(P(x) \rightarrow Q(x))$	Premise
②		$\exists x\,P(x)$	Premise
③	x_0	$P(x_0)$	Assumption
④		$P(x_0) \rightarrow Q(x_0)$	$\forall$x e 1
⑤		$Q(x_0)$	$\rightarrow$e 4,3
⑥		$Q(x_0)$	$\exists$x e 2,3-5
⑦		$\exists x\,Q(x)$	$\exists$x i 6

What is wrong with this proof?

$\forall x(Q(x)\rightarrow R(x))$, $\exists x(P(x)\wedge Q(x)) \vdash \exists x(P(x)\wedge R(x))$

- This formula may read as follows:
  - If all quakers (Q(x)) are reformists (R(x)) and if there is a protestant (P(x)) who is also a quaker, then there must be a protestant who is also a reformist

① $\forall x(Q(x)\rightarrow R(x))$ premise

② $\exists x(P(x)\wedge Q(x))$ premise

③

④

⑤

⑥

⑦

⑧

⑨

⑩ $\exists x\ (P(x) \wedge R(x))$

$\exists x\, P(x), \forall xy\, (P(x) \to Q(y)) \vdash \forall y\, Q(y)$

①	$\exists x\, P(x)$	premise
②	$\forall x \forall y\, (P(x) \to Q(x))$	premise
③	y_0	
④	$x_0\ P(x_0)$	assumption
⑤	$\forall y\, P(x_0) \to Q(y)$	$\forall$x e 2
⑥	$P(x_0) \to Q(y_0)$	$\forall$y e 5
⑦	$Q(y_0)$	$\to$ e 6,4
⑧	$Q(y_0)$	$\exists$x e 1,4-7
⑨	$\forall y\, Q(y)$	$\forall$ y i 3-8

$\exists x\, P(x),\ \forall x\, (P(x) \to Q(x)) \vdash \forall y\, Q(y)$???

①	$\exists x\, P(x)$	premise
②	$\forall x\, (P(x) \to Q(x))$	premise
③	x_0	
④	x_0 $P(x_0)$	assumption
⑤	$P(x_0) \to Q(x\_0)$	$\forall x$ e 2
⑥	$Q(x_0)$	$\to$ e 5,4
⑦	$Q(x_0)$	$\exists x$ e 1,4-6
⑧	$\forall y\, Q(y)$	$\forall$ y i 3-7

/* Compare with
$\exists x\, P(x),\ \forall xy\, (P(x) \to Q(y))$
$\vdash \forall y\, Q(y)$
***/**

$\neg \forall x\ \phi \vdash \exists x\ \neg\phi$

1	$\neg\forall x\, P(x)$	premise
2	$\neg\exists x\, \neg P(x)$	assumption
3	x_0	
4	$\neg P(x_0)$	assumption
5	$\exists x \neg P(x)$	$\exists x$ i 4
6	$\bot$	$\neg$e 5, 2
7	$P(x_0)$	RAA 4–6
8	$\forall x\, P(x)$	$\forall x$ i 3–7
9	$\bot$	$\neg$e 8, 1
10	$\exists x\, \neg P(x)$	RAA 2–9

1	$\neg\forall x\, \phi$	premise
2	$\neg\exists x\, \neg\phi$	assumption
3	x_0	
4	$\neg\phi[x_0/x]$	assumption
5	$\exists x \neg\phi$	$\exists x$ i 4
6	$\bot$	$\neg$e 5, 2
7	$\phi[x_0/x]$	RAA 4–6
8	$\forall x\, \phi$	$\forall x$ i 3–7
9	$\bot$	$\neg$e 8, 1
10	$\exists x\, \neg\phi$	RAA 2–9

$\neg \forall \mathbf{x}\ \phi \dashv \exists \mathbf{x}\ \neg \phi$

1	$\exists x\, \neg\phi$	assumption
2	$\forall x\, \phi$	assumption
3	x_0	
4	$\neg\phi[x_0/x]$	assumption
5	$\phi[x_0/x]$	$\forall x$ e 2
6	$\bot$	$\neg$e 5, 4
7	$\bot$	$\exists x$ e 1, 3–6
8	$\neg\forall x\, \phi$	$\neg$i 2–7

Comparison between Semantic tableau $\forall x\,(p(x) \rightarrow q(x)) \vdash (\forall x\, p(x) \rightarrow \forall x q(x))$

- $\forall x\,(p(x) \rightarrow q(x)) \vdash (\forall x\, p(x) \rightarrow \forall x q(x))$
- $\vdash \forall x\,(p(x) \rightarrow q(x)) \rightarrow (\forall x\, p(x) \rightarrow \forall x q(x))$

$\neg (\forall x\,(p(x) \rightarrow q(x)) \rightarrow (\forall x\, p(x) \rightarrow \forall x\, q(x))))$

↓

$\forall x\,(p(x) \rightarrow q(x)),\ \neg(\forall x\, p(x) \rightarrow \forall x\, q(x))))$

↓

$\forall x\,(p(x) \rightarrow q(x)),\ \forall x\, p(x),\ \neg \forall x q(x)$

↓

$\forall x\,(p(x) \rightarrow q(x)),\ \forall x\, p(x),\ \neg q(a)$

↓

$\forall x\,(p(x) \rightarrow q(x)),\ p(b),\ \neg q(a)$

↓

$p(b) \rightarrow q(a),\ p(b),\ \neg q(a)$

↙ ↘

$\neg\, p(b),\ p(b),\ \neg q(a)$ — **X**

$q(a),\ p(b),\ \neg q(b)$ — **X**

- $\forall x\,(p(x) \rightarrow q(x)) \vdash (\forall x\, p(x) \rightarrow \forall x q(x))$

①		$\forall x\,(p(x) \rightarrow q(x))$	Premise
②		$\forall x\,(p(x))$	assumption
③	x_0	$p(x_0) \rightarrow q(x_0)$	$\forall x$ e 1
④		$p(x_0)$	$\forall x$ e 2
⑤		$q(x_0)$	$\rightarrow$ e 2,3
⑥		$\forall x\, q(x)$	$\forall x$ i 3-5
⑦		$\forall x\, p(x) \rightarrow \forall x q(x)$	$\rightarrow$i 2-6