

Propositional Calculus - *Deductive Systems*

Moonzoo Kim

CS Division of EECS Dept.
KAIST

moonzoo@cs.kaist.ac.kr

<http://pswlab.kaist.ac.kr/courses/cs402-07>

Deductive proofs (1/3)

- Suppose we want to know if ϕ belongs to the theory $\mathcal{T}(U)$.
 - By Thm 2.38 $U \models \phi$ iff $\models A_1 \wedge \dots \wedge A_n \rightarrow \phi$ where $U = \{A_1, \dots, A_n\}$
 - Thus, $\phi \in \mathcal{T}(U)$ iff a decision procedure for validity answers 'yes'
- However, there are several problems with this **semantic** approach
 - The set of axioms may be **infinite**
 - e.x. Hilbert deductive system \mathcal{H} has an **axiom schema** $(A \rightarrow (B \rightarrow A))$, which generates an infinite number of axioms by replacing schemata variables A, B and C with infinitely many subformulas (e.g. $\phi \wedge \psi, \neg \phi \vee \psi$, etc)
 - e.x.2. Peano and ZFC theories cannot be finitely axiomatized.
 - Very few logics have **decision procedures** for validity of ϕ
 - ex. propositional logic has a decision procedure using truth table
 - ex2. predicate logic does **not** have such decision procedure
- There is another approach to logic called **deductive proofs**.
 - Instead of working with semantic concepts like **interpretation/model** and **consequence**
 - we choose a set of **axioms** and a set of **syntactical rules** for deducing new formulas from the axioms

Deductive proofs (2/3)

Def 3.1

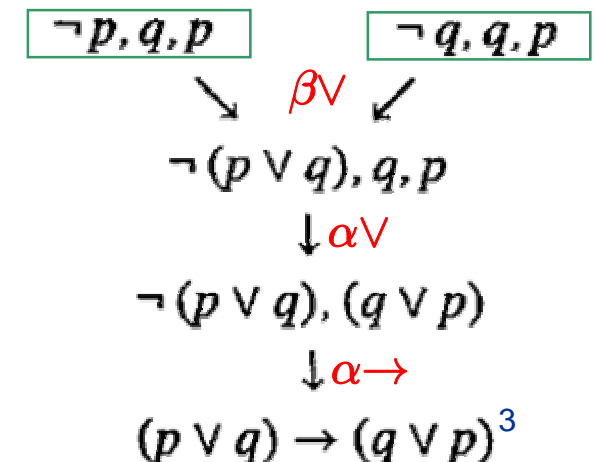
- A **deductive system** consists of
 - a set of **axioms** and
 - a set of **inference rules**
- A **proof** in a deductive system is a **sequence of sets of formulas** s.t. each element is either an **axiom** or it can be inferred from previous elements of the sequence using a rule of inference
- If $\{A\}$ is the last element of the sequence, A is a **theorem**, the sequence is a proof of A , and A is provable, denoted $\vdash A$

Example of a proof of $(p \vee q) \rightarrow (q \vee p)$ in gentzen system \mathcal{G}

- $\{\neg p, q, p\} \cdot \{\neg q, q, p\} \cdot \{\neg(p \vee q), q, p\} \cdot \{\neg(p \vee q), (q \vee p)\} \cdot \{(p \vee q) \rightarrow (q \vee p)\}$

axioms

theorem



- tree representation of this proof is more intuitive

Deductive proofs (3/3)

- Deductive proofs has following benefits
 - There may be an infinite number of axioms, but only a **finite number of axioms** will appear in any proof
 - Any particular proof consists of a finite sequence of sets of formulas, and the **legality of each individual deduction** can be easily and efficiently determined from the **syntax** of the formulas
 - The proof of a formula clearly shows which axioms, theorems and rules are used and for what purposes.
 - Such a **pattern** (i.e. relationship between formulas) can then be transferred to other similar proofs, or modified to prove different results.
 - Lemmas and corollaries can be obtained and can be used later
- But with a new problem
 - deduction is defined purely in terms of syntactical formula manipulation
 - But it is **not** amenable to systematic search procedures
 - no brute-force search is possible because any axiom can be used

The Gentzen system \mathcal{G}

- Def 3.2 The Gentzen system \mathcal{G} is a deductive system.
 - The **axioms** are the sets of formulas containing a **complementary pairs of literals**
 - ex. $\{\neg p, p, p \wedge q\}$ can be an axiom, but $\{\neg p, q, p \wedge q\}$ is not.
 - The **rules of inferences** are:
 - note that a set of formulas in \mathcal{G} is an implicit **disjunction**

premise $\frac{\vdash U_1 \cup \{\alpha_1, \alpha_2\}}{\vdash U_1 \cup \{\alpha\}}$

conclusion

$\frac{\vdash U_1 \cup \{\beta_1\} \quad \vdash U_2 \cup \{\beta_2\}}{\vdash U_1 \cup U_2 \cup \{\beta\}}$

8 α -rules

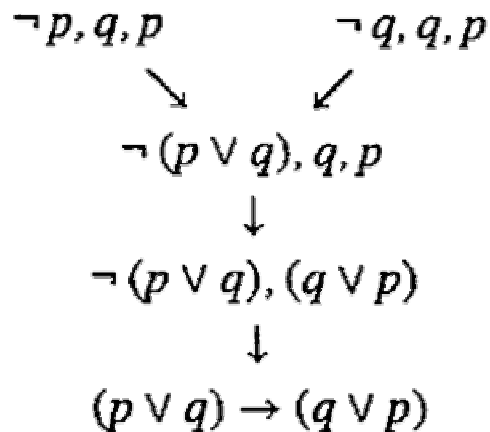
α	α_1	α_2
$\neg\neg A$	A	
$\neg(A_1 \wedge A_2)$	$\neg A_1$	$\neg A_2$
$A_1 \vee A_2$	A_1	A_2
$A_1 \rightarrow A_2$	$\neg A_1$	A_2
$A_1 \uparrow A_2$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \downarrow A_2)$	A_1	A_2
$\neg(A_1 \leftrightarrow A_2)$	$\neg(A_1 \rightarrow A_2)$	$\neg(A_2 \rightarrow A_1)$
$A_1 \oplus A_2$	$\neg(A_1 \rightarrow A_2)$	$\neg(A_2 \rightarrow A_1)$

7 β -rules

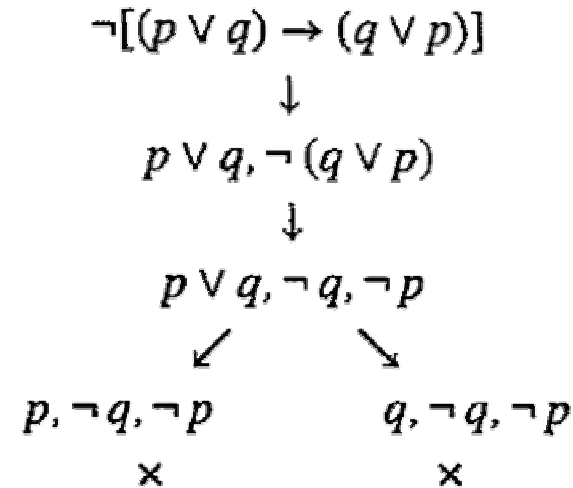
β	β_1	β_2
$B_1 \wedge B_2$	B_1	B_2
$\neg(B_1 \vee B_2)$	$\neg B_1$	$\neg B_2$
$\neg(B_1 \rightarrow B_2)$	B_1	$\neg B_2$
$\neg(B_1 \uparrow B_2)$	B_1	B_2
$B_1 \downarrow B_2$	$\neg B_1$	$\neg B_2$
$B_1 \leftrightarrow B_2$	$B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$
$\neg(B_1 \oplus B_2)$	$B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$

Soundness and completeness of \mathcal{G}

- Note that there are close relationship between a deductive proof of ϕ and semantic tableau of ϕ



A proof in \mathcal{G}



Semantic tableau

Soundness and completeness of \mathcal{G}

- Thm 3.6 Let U be a set of formulas and \bar{U} be the set of complements of formulas in U . Then $\vdash U$ in \mathcal{G} iff there is a closed semantic tableau T for \bar{U}
- Proof of completeness,
 - $\vdash U$ in \mathcal{G} if there exists a closed T for \bar{U} exists
 - induction on the height of T , h
 - $h=0$
 - T consists of a single node labeled by \bar{U} , a set of literals containing a complementary pair (say $\{p, \neg p\}$), that is $\bar{U} = \bar{U}_0 \cup \{p, \neg p\}$
 - Obviously $U = U_0 \cup \{\neg p, p\}$ is an axiom in \mathcal{G} , hence $\vdash U$

Soundness and completeness of \mathcal{G}

■ Proof of completeness (continued)

■ $\vdash U$ in \mathcal{G} if there exists a closed T for \bar{U} exists

■ $h > 0$

■ Some tableau α or β rule was used at the root n of T on a formula $\bar{A} \in \bar{U}$, that is $\bar{U} = \bar{U}_0 \cup \{\bar{A}\}$

■ Case of α rule

■ A tableau α -rule was used on (a formula such as) $\bar{A} = \neg (A_1 \vee A_2)$ to produce the node n' labeled $\bar{U}' = \bar{U}_0' \cup \{\neg A_1, \neg A_2\}$. The subtree rooted at n' is a closed tableau for \bar{U}' , so by the inductive hypothesis, $\vdash U_0' \cup \{A_1, A_2\}$. Using the α -rule in \mathcal{G} , $\vdash U_0 \cup \{A_1 \vee A_2\}$, that is $\vdash U$

■ Case of β rule

■ A tableau β -rule was used on (a formula such as) $\bar{A} = \neg (A_1 \wedge A_2)$ to produce the node n' and n'' labeled $\bar{U}' = \bar{U}_0 \cup \{\neg A_1\}$, $\bar{U}'' = \bar{U}_0 \cup \{\neg A_2\}$, respectively. By the inductive hypothesis, $\vdash U_0 \cup \{A_1\}$ and $\vdash U_0 \cup \{A_2\}$. Using the β -rule in \mathcal{G} , $\vdash U_0 \cup \{A_1 \wedge A_2\}$, that is $\vdash U$