

Propositional Calculus

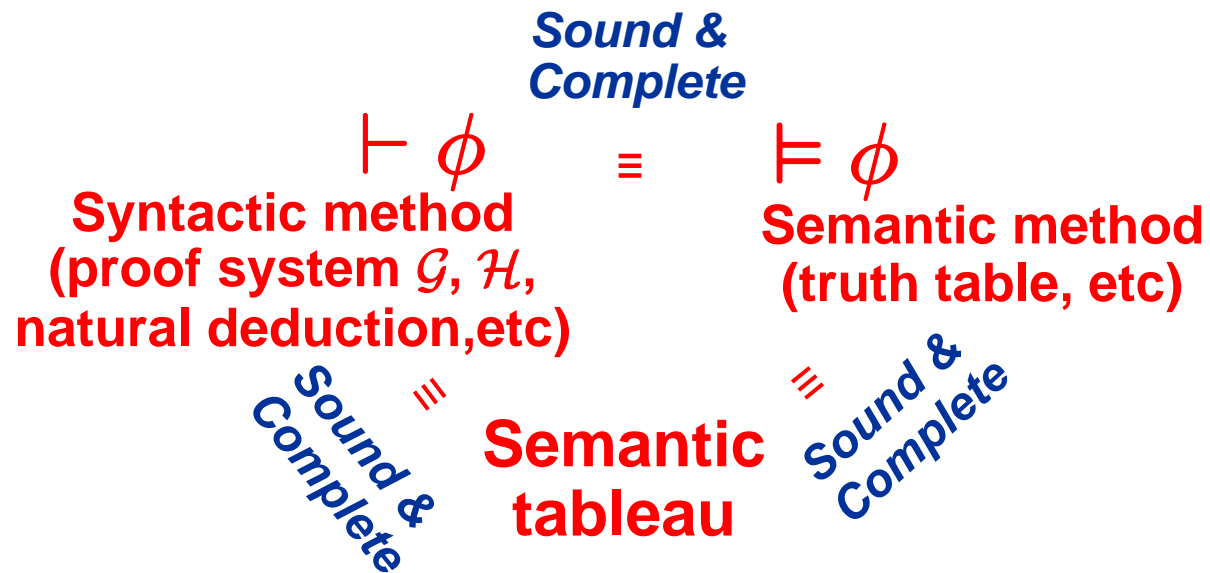
- *Soundness & Completeness of \mathcal{H}*

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Review

- Goal of logic
 - To check whether given a formula ϕ is valid
 - To prove a given formula ϕ



Soundness of \mathcal{H} (1/2)

- Thm 3.34 \mathcal{H} is sound, that is $\vdash A$ then $\models A$
 - Proof is by structural induction
 - We show that
 1. the all three axioms are valid and that
 2. if the premises of MP are valid, so is the conclusion
- Task 1: to prove \models Axiom1, \models Axiom2, and \models Axiom3
 - By showing the semantic tableau of the negated axiom is closed

$$\begin{array}{c}
 \neg[A \rightarrow (B \rightarrow A)] \\
 \downarrow \\
 A, \neg(B \rightarrow A) \\
 \downarrow \\
 A, B, \neg A \\
 \times
 \end{array}$$

$$\begin{array}{c}
 \neg[(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)] \\
 \downarrow \\
 \neg B \rightarrow \neg A, \neg(A \rightarrow B) \\
 \downarrow \\
 \neg B \rightarrow \neg A, A, \neg B \\
 \swarrow \quad \searrow \\
 \neg\neg B, A, \neg B \qquad \neg A, A, \neg B \\
 \downarrow \qquad \qquad \times \\
 B, A, \neg B \\
 \times
 \end{array}$$

Soundness of \mathcal{H} (2/2)

- Task 2: proof by RAA (귀류법)
 - Suppose that MP were not sound.
 - Then there would be a set of formulas $\{A, A \rightarrow B, B\}$ such that A and $A \rightarrow B$ are valid, but B is not valid

$$\frac{\vdash A \quad \vdash A \rightarrow B}{\vdash B}$$

- If B is not valid, there is an interpretation v such that $v(B) = F$. Since A and $A \rightarrow B$ are valid, for **any** interpretation, in particular for v , $v(A) = v(A \rightarrow B) = T$. From this we deduce that $v(B) = T$ contradicting the choice of v

Completeness of $\mathcal{H}(1/4)$

- Thm 3.35 \mathcal{H} is complete, that is, if $\models A$ then $\vdash A$
- Any valid formula can be proved in \mathcal{G} (thm 3.8). We will show how a proof in \mathcal{G} can be mechanically transformed into a proof in \mathcal{H}
- The exact correspondence is that if the **set** of formulas U is provable in \mathcal{G} then the single formula $\bigvee U$ is provable in \mathcal{H}
 - A problem is that
 - We can show that $\{\neg p, p\}$ is an axiom in \mathcal{G} then $\vdash p \vee \neg p$ in \mathcal{H} since this is simply Thm 3.10 ($\vdash A \rightarrow A$)
 - Note that $A \vee B$ is an abbreviation for $\neg A \rightarrow B$
 - Similarly $A \wedge B$ is an abbreviation for $\neg(A \rightarrow \neg B)$
 - But if the axiom in \mathcal{G} is $\{q, \neg p, r, p, s\}$, we **cannot** immediately conclude that $\vdash q \vee \neg p \vee r \vee p \vee s$

Completeness of $\mathcal{H}(2/4)$

- Lem 3.36 If $U' \subseteq U$ and $\vdash \bigvee U'$ (in \mathcal{H}) then $\vdash \bigvee U$ (in \mathcal{H})
- The proof is by induction using Thm 3.31 through 3.33
 - Suppose we have a proof of $\bigvee U'$. By repeated application of Thm 3.31, we can transform this into a proof of $\bigvee U''$, where U'' is a permutation of the elements of U .
 - Thm 3.31 Weakening $\vdash A \rightarrow A \vee B$ and $\vdash B \rightarrow A \vee B$
 - Now by repeated applications of the commutativity and associativity of disjunction, we can move the elements of U'' to their proper places
 - Thm 3.32 Commutativity rule: $\vdash A \vee B \leftrightarrow B \vee A$
 - Thm 3.33 Associativity rule : $\vdash A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C$

Completeness of $\mathcal{H}(3/4)$

- Completeness proof by induction on the **structure of the proof** in \mathcal{G}
 - We are transforming a proof in \mathcal{G} to a proof in \mathcal{H}
- Task 1:
 - If U is an **axiom**, it contains a pair of complementary literals and $\vdash \neg p \vee p$ can be proved in \mathcal{H} . BY Lem 3.36, this may be transformed into a proof of $\vee U$.
 - Lem 3.36 If $U' \subseteq U$ and $\vdash \vee U'$ (in \mathcal{H}) then $\vdash \vee U$ (in \mathcal{H})

Completeness of $\mathcal{H}(4/4)$

■ Task 2:

- The last step in the proof of U in \mathcal{G} is the application of an α or β rule.

- Case 1: An α rule was used in \mathcal{G} to infer
$$\frac{\vdash U_1 U \{A_1, A_2\}}{\vdash U_1 U \{A_1 \vee A_2\}}$$

- By the inductive hypothesis, $\vdash (\vee U_1 \vee A_1) \vee A_2$ in \mathcal{H} from which we infer $\vdash \vee U_1 \vee (A_1 \vee A_2)$ by associativity

- Case 2: An β rule was used in \mathcal{G} to infer

$$\frac{\vdash U_1 U \{A_1\} \quad \vdash U_2 U \{A_2\}}{\vdash U_1 U U_2 U \{A_1 \wedge A_2\}}$$

- By the inductive hypothesis, $\vdash \vee U_1 \vee A_1$ and $\vdash \vee U_2 \vee A_2$ in \mathcal{H} . Thus $\vdash \vee U_1 \vee \vee U_2 \vee (A_1 \wedge A_2)$

Consistency

- Def 3.38 A set of formulas U is inconsistent iff for some formula A , $U \vdash A$ and $U \vdash \neg A$. U is consistent iff it is not inconsistent
- Thm 3.39 U is inconsistent iff for all A , $U \vdash A$
 - Proof: Let A be an arbitrary formula. Since U is inconsistent, for some formula B , $U \vdash B$ and $U \vdash \neg B$.
 - By Thm 3.21 $\vdash B \rightarrow (\neg B \rightarrow A)$. Using MP twice, $U \vdash A$.
- Corollary 3.40 U is consistent iff for some A , $U \not\vdash A$
- Thm 3.41 $U \vdash A$ iff $U \cup \{\neg A\}$ is inconsistent

Variants of \mathcal{H} (1/2)

- Variant Hilbert systems almost invariably have MP as the only rule. They differ in the choice of primitive operators and axioms
- \mathcal{H}' replace Axiom 3 by
 - Axiom 3' $\vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
- Thm 3.44 \mathcal{H} and \mathcal{H}' are equivalent

- A proof of Axiom 3' in \mathcal{H}
- The other direction?

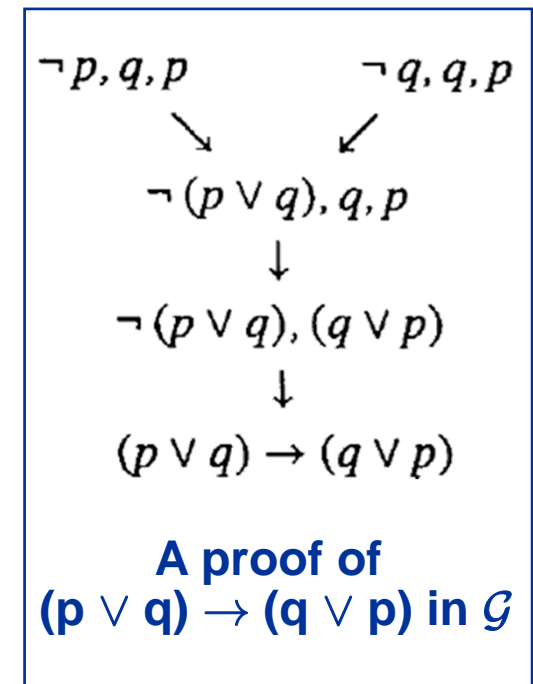
1.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B$	Assumption
2.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow A$	Assumption
3.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A$	MP 1,2
4.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow \neg A$	Assumption
5.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A \rightarrow B$	Contrapositive 4
6.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash B$	MP 3,5
7.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow B$	Deduction 7
8.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash (\neg B \rightarrow B) \rightarrow B$	Theorem 3.29
9.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash B$	MP 8,9
10.	$\{\neg B \rightarrow \neg A\} \vdash (\neg B \rightarrow A) \rightarrow B$	Deduction 9
11.	$\vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$	Deduction 10

Variants of \mathcal{H} (2/2)

- \mathcal{H}'' has the same MP rule but a set of axioms as
 - Axiom 1 $\vdash A \vee A \rightarrow A$
 - Axiom 2 $\vdash A \rightarrow A \vee B$
 - Axiom 3 $\vdash A \vee B \rightarrow B \vee A$
 - Axiom 4 $\vdash B \rightarrow C \rightarrow (A \vee B \rightarrow A \vee C)$
 - Note that it is also possible to consider \vee as the primitive binary operator. Then, \rightarrow is defined by $\neg A \vee B$.
- Yet another variant of Hilbert system \mathcal{H}''' has only one axiom with MP
 - Meredith's axiom
 - $(\{[A \rightarrow B] \rightarrow (\neg C \rightarrow \neg D)\} \rightarrow C) \rightarrow E \rightarrow [(E \rightarrow A) \rightarrow (D \rightarrow A)]$

Subformula property

- Def 3.48 A deductive system has the **subformula property** if any formula appearing in a proof of A is either a subformula of A or the negation of a subformula of A
- \mathcal{G} has the subformula property while \mathcal{H} obviously does not since MP 'erase' formulas
 - That is why a proof in \mathcal{H} is harder than a proof in \mathcal{G}
- If a deductive system has the subformula property, then **mechanical proof** may be possible since there exists only
 - there exist only a **finite number of subformulas** for a finite formula ϕ
 - there exist only a **finite number of inference rules**



Automated proof

- One desirable property of a deductive system is to generate an **automated/mechanical proof**
 - We have **decision procedure** to check validity of a propositional formula automatically (i.e., truth table and semantic tableau)
 - Note that decision procedure requires knowledge on **all interpretations** (i.e., infinite number of interpretations in general) which is not feasible except propositional logic
- A deductive proof requires only a **finite set of sets of formulas**, because a deductive proof system analyzes the target formula only, not its interpretations.
 - Many research works to develop **(semi)automated theorem prover**
- No obvious connection between the formula and its proof in \mathcal{H} makes a proof in \mathcal{H} difficult (‘ no mechanical proof)
 - A human being has to rely on his/her brain to select proper axioms