

Predicate Calculus

- Natural deduction (2/2)

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Summary of proof rules of natural deduction

	introduction	elimination
\wedge	$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\phi \wedge \psi}{\psi} \wedge e_2$
\vee	$\frac{\phi}{\phi \vee \psi} \vee i_1 \quad \frac{\psi}{\phi \vee \psi} \vee i_2$	$\frac{\phi \vee \psi \quad \begin{array}{ c } \hline \phi \\ \hline \vdots \\ \hline \chi \\ \hline \end{array} \quad \begin{array}{ c } \hline \psi \\ \hline \vdots \\ \hline \chi \\ \hline \end{array}}{\chi} \vee e$
\rightarrow	$\frac{\begin{array}{ c } \hline \phi \\ \hline \vdots \\ \hline \psi \\ \hline \end{array}}{\phi \rightarrow \psi} \rightarrow i$	$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e$
\neg	$\frac{\begin{array}{ c } \hline \phi \\ \hline \vdots \\ \hline \perp \\ \hline \end{array}}{\neg \phi} \neg i$	$\frac{\phi \quad \neg \phi}{\perp} \neg e$
\perp	(no introduction rule for \perp)	$\frac{\perp}{\phi} \perp e$
$\neg\neg$		$\frac{\neg\neg\phi}{\phi} \neg\neg e$

assumption ϕ ψ

assumption ϕ

assumption ϕ

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi} MT$$

$$\frac{\phi}{\neg\neg\phi} \neg\neg i$$

assumption $\neg\phi$

$$\frac{\begin{array}{|c|} \hline \neg\phi \\ \hline \vdots \\ \hline \perp \\ \hline \end{array}}{\phi} RAA$$

$$\frac{}{\phi \vee \neg\phi} LEM$$

$$\frac{\begin{array}{|c|} \hline x_0 \\ \hline \vdots \\ \hline \phi[x_0/x] \\ \hline \end{array}}{\forall x \phi} \forall x i$$

$$\frac{\forall x \phi}{\phi[t/x]} \forall x e$$

$$\frac{\phi[t/x]}{\exists x \phi} \exists x i$$

assumption $\phi[x_0/x]$

$$\frac{\exists x \phi \quad \begin{array}{|c|} \hline x_0 \quad \phi[x_0/x] \\ \hline \vdots \\ \hline \chi \\ \hline \end{array}}{\chi} \exists x e$$

Example

$\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$

- | | | |
|---|-------------------------------------|---------------------|
| ① | $\forall x (P(x) \rightarrow Q(x))$ | Premise |
| ② | $\exists x P(x)$ | Premise |
| ③ | $x_0 P(x_0)$ | Assumption |
| ④ | $P(x_0) \rightarrow Q(x_0)$ | $\forall x e 1$ |
| ⑤ | $Q(x_0)$ | $\rightarrow e 4,3$ |
| ⑥ | $\exists x Q(x)$ | $\exists x i 5$ |
| ⑦ | $\exists x Q(x)$ | $\exists x e 2,3-6$ |

$\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$

- | | | |
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| ③ | $x_0 P(x_0)$ | Assumption |
| ④ | $P(x_0) \rightarrow Q(x_0)$ | $\forall x e 1$ |
| ⑤ | $Q(x_0)$ | $\rightarrow e 4,3$ |
| ⑥ | $Q(x_0)$ | $\exists x e 2,3-5$ |
| ⑦ | $\exists x Q(x)$ | $\exists x i 6$ |

What is wrong with this proof?

$$\forall x(Q(x) \rightarrow R(x)), \exists x(P(x) \wedge Q(x)) \vdash \exists x(P(x) \wedge R(x))$$

- This formula may read as follows:
 - If all quakers ($Q(x)$) are reformists ($R(x)$) and if there is a protestant ($P(x)$) who is also a quaker, **then** there must be a protestant who is also a reformist

① $\forall x(Q(x) \rightarrow R(x))$ premise

② $\exists x(P(x) \wedge Q(x))$ premise

③

④

⑤

⑥

⑦

⑧

⑨

⑩ $\exists x (P(x) \wedge R(x))$

$$\exists x P(x), \forall xy (P(x) \rightarrow Q(y)) \vdash \forall y Q(y)$$

- | | | |
|---|---|---------------------|
| ① | $\exists x P(x)$ | premise |
| ② | $\forall x \forall y (P(x) \rightarrow Q(x))$ | premise |
| ③ | y_0 | |
| ④ | $x_0 P(x_0)$ | assumption |
| ⑤ | $\forall y P(x_0) \rightarrow Q(y)$ | $\forall x$ e 2 |
| ⑥ | $P(x_0) \rightarrow Q(y_0)$ | $\forall y$ e 5 |
| ⑦ | $Q(y_0)$ | \rightarrow e 6,4 |
| ⑧ | $Q(y_0)$ | $\exists x$ e 1,4-7 |
| ⑨ | $\forall y Q(y)$ | $\forall y$ i 3-8 |

$\exists x P(x), \forall x (P(x) \rightarrow Q(x)) \vdash \forall y Q(y) ???$

①	$\exists x P(x)$	premise
②	$\forall x (P(x) \rightarrow Q(x))$	premise
③	x_0	
④	$x_0 P(x_0)$	assumption
⑤	$P(x_0) \rightarrow Q(x_0)$	$\forall x e 2$
⑥	$Q(x_0)$	$\rightarrow e 5,4$
⑦	$Q(x_0)$	$\exists x e 1,4-6$
⑧	$\forall y Q(y)$	$\forall y i 3-7$

/* Compare with
 $\exists x P(x), \forall xy (P(x) \rightarrow Q(y))$
 $\vdash \forall y Q(y)$
***/**

$$\neg \forall x \phi \vdash \exists x \neg \phi$$

1	$\neg \forall x P(x)$	premise	
2	$\neg \exists x \neg P(x)$ assumption		
3	x_0		
4	$\neg P(x_0)$ assumption		
5	$\exists x \neg P(x)$ $\exists x$ i 4		
6	\perp $\neg e$ 5, 2		
7	$P(x_0)$ RAA 4–6		
8	$\forall x P(x)$ $\forall x$ i 3–7		
9	\perp $\neg e$ 8, 1		
10	$\exists x \neg P(x)$ RAA 2–9		

1	$\neg \forall x \phi$	premise	
2	$\neg \exists x \neg \phi$ assumption		
3	x_0		
4	$\neg \phi[x_0/x]$ assumption		
5	$\exists x \neg \phi$ $\exists x$ i 4		
6	\perp $\neg e$ 5, 2		
7	$\phi[x_0/x]$ RAA 4–6		
8	$\forall x \phi$ $\forall x$ i 3–7		
9	\perp $\neg e$ 8, 1		
10	$\exists x \neg \phi$ RAA 2–9		

$$\neg \forall x \phi \vdash \exists x \neg \phi$$

1	$\exists x \neg \phi$	assumption
2	$\forall x \phi$	assumption
3	x_0	
4	$\neg \phi[x_0/x]$	assumption
5	$\phi[x_0/x]$	$\forall x e 2$
6	\perp	$\neg e 5, 4$
7	\perp	$\exists x e 1, 3-6$
8	$\neg \forall x \phi$	$\neg i 2-7$

Comparison between Semantic tableau

$$\forall x (p(x) \rightarrow q(x)) \vdash (\forall x p(x) \rightarrow \forall x q(x))$$

■ $\forall x (p(x) \rightarrow q(x)) \vdash (\forall x p(x) \rightarrow \forall x q(x))$ ■

■ $\vdash \forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$

$$\neg (\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x)))$$

$$\forall x (p(x) \rightarrow q(x)), \neg (\forall x p(x) \rightarrow \forall x q(x))$$

$$\forall x (p(x) \rightarrow q(x)), \forall x p(x), \neg \forall x q(x)$$

$$\forall x (p(x) \rightarrow q(x)), \forall x p(x), \neg q(a)$$

$$\forall x (p(x) \rightarrow q(x)), p(b), \neg q(a)$$

$$p(b) \rightarrow q(a), p(b), \neg q(a)$$

$$\begin{array}{cc} \neg p(b), p(b), \neg q(a) & q(a), p(b), \neg q(b) \\ \mathbf{x} & \mathbf{x} \end{array}$$

$$\forall x (p(x) \rightarrow q(x)) \vdash (\forall x p(x) \rightarrow \forall x q(x))$$

① $\forall x (p(x) \rightarrow q(x))$ Premise

② $\forall x (p(x))$ assumption

③ x_0 $p(x_0) \rightarrow q(x_0)$ $\forall x$ e 1

④ $p(x_0)$ $\forall x$ e 2

⑤ $q(x_0)$ \rightarrow e 2,3

⑥ $\forall x q(x)$ $\forall x$ i 3-5

⑦ $\forall x p(x) \rightarrow \forall x q(x)$ \rightarrow i 2-6