Propositional Calculus
- Deductive Systems

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Deductive proofs (1/3)

- Suppose we want to know if $\phi$ belongs to the theory $\mathcal{T}(U)$.
  - By Thm 2.38 $U \models \phi$ iff $\models A_1 \land \ldots \land A_n \rightarrow \phi$ where $U = \{ A_1, \ldots, A_n \}$
  - Thus, $\phi \in \mathcal{T}(U)$ iff a decision procedure for validity answers ‘yes’

- However, there are several problems with this semantic approach
  - The set of axioms may be infinite
    - e.x. Hilbert deductive system $\mathcal{H}$ has an axiom schema $(A \rightarrow (B \rightarrow A))$, which generates an infinite number of axioms by replacing schemata variables $A, B$ and $C$ with infinitely many subformulas (e.g. $\phi \land \psi, \neg \phi \lor \psi$, etc)
    - e.x. 2. Peano and ZFC theories cannot be finitely axiomatized.
  - Very few logics have decision procedures for validity of $\phi$
    - e.x. propositional logic has a decision procedure using truth table
    - e.x2. predicate logic does not have such decision procedure

- There is another approach to logic called deductive proofs.
  - Instead of working with semantic concepts like interpretation/model and consequence
  - we choose a set of axioms and a set of syntactical rules for deducing new formulas from the axioms
Def 3.1

A deductive system consists of
- a set of axioms and
- a set of inference rules

A proof in a deductive system is a sequence of sets of formulas s.t. each element is either an axiom or it can be inferred from previous elements of the sequence using a rule of inference.

If \{A\} is the last element of the sequence, A is a theorem, the sequence is a proof of A, and A is provable, denoted \( \vdash A \)

Example of a proof of \((p \lor q) \rightarrow (q \lor p)\) in gentzen system \(G\)

\[
\begin{align*}
\{\neg p, q, p\}.\{\neg q, q, p\}.\{\neg(p \lor q), q, p\}.\{\neg(p \lor q), (q \lor p)\}.\{(p \lor q) \rightarrow (q \lor p)\}
\end{align*}
\]

axioms: 
- tree representation of this proof is more intuitive
Deductive proofs has following benefits

- There may be an infinite number of axioms, but only a finite number of axioms will appear in any proof.
- Any particular proof consists of a finite sequence of sets of formulas, and the legality of each individual deduction can be easily and efficiently determined from the syntax of the formulas.
- The proof of a formula clearly shows which axioms, theorems and rules are used and for what purposes.
  - Such a pattern (i.e. relationship between formulas) can then be transferred to other similar proofs, or modified to prove different results.
  - Lemmas and corollaries can be obtained and can be used later.

But with a new problem

- Deduction is defined purely in terms of syntactical formula manipulation.
- But it is not amenable to systematic search procedures.
  - No brute-force search is possible because any axiom can be used.
Def 3.2 The Gentzen system $\mathcal{G}$ is a deductive system.

- The axioms are the sets of formulas containing a complementary pairs of literals.
  - ex. \( \{ \neg p, p, p \land q \} \) can be an axiom, but \( \{ \neg p, q, p \land q \} \) is not.
- The rules of inferences are:
  - note that a set of formulas in $\mathcal{G}$ is an implicit disjunction.

\[
\begin{array}{c}
\text{premise} \\
\vdash U_1 \cup \{ \alpha_1, \alpha_2 \}
\end{array}
\quad
\begin{array}{c}
\text{conclusion} \\
\vdash U_1 \cup \{ \alpha \}
\end{array}
\]

\[
\begin{array}{c}
\text{premise} \\
\vdash U_1 \cup \{ \beta_1 \} \\
\vdash U_2 \cup \{ \beta_2 \}
\end{array}
\quad
\begin{array}{c}
\text{conclusion} \\
\vdash U_1 \cup U_2 \cup \{ \beta \}
\end{array}
\]

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<th>$\alpha$</th>
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<th>$\alpha_2$</th>
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<tbody>
<tr>
<td>$\neg \neg A$</td>
<td>$A$</td>
<td></td>
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<tr>
<td>$\neg (A_1 \land A_2)$</td>
<td>$\neg A_1$</td>
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</tr>
<tr>
<td>$A_1 \uparrow A_2$</td>
<td>$\neg A_1$</td>
<td>$\neg A_2$</td>
</tr>
<tr>
<td>$\neg (A_1 \downarrow A_2)$</td>
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</tr>
<tr>
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<td>$\neg (A_1 \rightarrow A_2)$</td>
<td>$\neg (A_2 \rightarrow A_1)$</td>
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<tr>
<td>$A_1 \oplus A_2$</td>
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8 $\alpha$-rules

<table>
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<tr>
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<tr>
<td>$\neg (B_1 \oplus B_2)$</td>
<td>$B_1 \rightarrow B_2$</td>
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</tr>
</tbody>
</table>

7 $\beta$-rules
Soundness and completeness of $G$

- Note that there are close relationships between a deductive proof of $\phi$ and semantic tableau of $\phi$

A proof in $G$

\[
\neg p, q, p \quad \neg q, q, p
\]

\[
\downarrow \quad \checkmark
\]

\[
\neg (p \lor q), q, p
\]

\[
\downarrow
\]

\[
\neg (p \lor q), (q \lor p)
\]

\[
\downarrow
\]

\[
(p \lor q) \rightarrow (q \lor p)
\]

Semantic tableau

\[
\neg [(p \lor q) \rightarrow (q \lor p)]
\]

\[
\downarrow
\]

\[
p \lor q, \neg (q \lor p)
\]

\[
\downarrow
\]

\[
p \lor q, \neg q, \neg p
\]

\[
\checkmark \quad \checkmark
\]

\[
p, \neg q, \neg p
\]

\[
q, \neg q, \neg p
\]

\[
\times \quad \times
\]
Soundness and completeness of \( \mathcal{G} \)

- Thm 3.6 Let \( U \) be a set of formulas and \( \bar{U} \) be the set of complements of formulas in \( U \). Then \( \vdash \bar{U} \) in \( \mathcal{G} \) iff there is a closed semantic tableau \( T \) for \( \bar{U} \)

- Proof of completeness,
  - \( \vdash \bar{U} \) in \( \mathcal{G} \) if there exists a closed \( T \) for \( \bar{U} \) exists
  - induction on the height of \( T \), \( h \)
  - \( h=0 \)
    - \( T \) consists of a single node labeled by \( \bar{U} \), a set of literals containing a complementary pair (say \( \{p, \neg p\} \)), that is \( \bar{U} = \bar{U}_0 \cup \{p, \neg p\} \)
    - Obviously \( U = U_0 \cup \{\neg p, p\} \) is an axiom in \( \mathcal{G} \), hence \( \vdash U \)
Soundness and completeness of $G$

- **Proof of completeness (continued)**
  - $\vdash \tilde{U}$ in $G$ if there exists a closed $T$ for $\tilde{U}$ exists
  - $h > 0$
    - Some tableau $\alpha$ or $\beta$ rule was used at the root $n$ of $T$ on a formula $\tilde{A} \in \tilde{U}$, that is $\tilde{U} = \tilde{U}_0 \cup \{\tilde{A}\}$
    - **Case of $\alpha$ rule**
      - A tableau $\alpha$-rule was used on (a formula such as) $\tilde{A} = \neg (A_1 \lor A_2)$ to produce the node $n'$ labeled $\tilde{U}' = \tilde{U}_0' \cup \{\neg A_1, \neg A_2\}$. The subtree rooted at $n'$ is a closed tableau for $\tilde{U}'$, so by the inductive hypothesis, $\vdash \tilde{U}_0' \cup \{A_1, A_2\}$. Using the $\alpha$-rule in $G$, $\vdash \tilde{U}_0 \cup \{A_1 \lor A_2\}$, that is $\vdash \tilde{U}$
    - **Case of $\beta$ rule**
      - A tableau $\beta$-rule was used on (a formula such as) $\tilde{A} = \neg (A_1 \land A_2)$ to produce the node $n'$ and $n''$ labeled $\tilde{U}' = \tilde{U}_0' \cup \{\neg A_1\}$, $\tilde{U}'' = \tilde{U}_0'' \cup \{\neg A_2\}$, respectively. By the inductive hypothesis, $\vdash \tilde{U}_0 \cup \{A_1\}$ and $\vdash \tilde{U}_0 \cup \{A_2\}$. Using the $\beta$-rule in $G$, $\vdash \tilde{U}_0 \cup \{A_1 \land A_2\}$, that is $\vdash \tilde{U}$