First order theories

(Chapter 1, Sections 1.4 – 1.5)

From the slides for the book
“Decision procedures”
by D.Kroening and O.Strichman

First order logic

- A first order theory consists of
  - Variables
  - Logical symbols: $\wedge$ $\vee$ $\neg$ $\forall$ $\exists$ `(’ `)’
  - Non-logical Symbols $\sum$: Constants, predicate and function symbols
  - Syntax

Examples

- Σ = {0,1, ‘+’, ‘>’}
  - ‘0’,’1’ are constant symbols
  - ‘+’ is a binary function symbol
  - ‘>’ is a binary predicate symbol

- An example of a Σ-formula:

$$\exists y\ \forall x.\ x > y$$

Examples

- Σ = {1, '>', '<', 'isprime'}
  - '1' is a constant symbol
  - '>', '<' are binary predicates symbols
  - 'isprime' is a unary predicate symbol
- An example Σ-formula:

$$\forall n\ \exists p.\ n > 1 \rightarrow \text{isprime}(p) \wedge n < p < 2n.$$

- Are these formulas valid ?
- So far these are only symbols, strings. No meaning yet.

Interpretations

- Let Σ = {0,1, ‘+’, ‘=’} where 0,1 are constants, ‘+’ is a binary function symbol and ‘=’ a predicate symbol.
- Let $\phi = \exists x.\ x + 0 = 1$
- Q: Is ϕ true in $\mathcal{N}_0$?
- A: Depends on the interpretation!

Structures

- A structure is given by:
  1. A domain
  2. An interpretation of the nonlogical symbols: i.e.,
     - Maps each predicate symbol to a predicate of the same arity
     - Maps each function symbol to a function of the same arity
     - Maps each constant symbol to a domain element
  3. An assignment of a domain element to each free (unquantified) variable

Structures

- Remember $\phi = \exists x.\ x + 0 = 1$
- Consider the structure S:
  - Domain: $\mathcal{N}_0$
  - Interpretation:
    - '0' and '1' are mapped to 0 and 1 in $\mathcal{N}_0$
    - '=' $\mapsto$ = (equality)
    - '+' $\mapsto$ * (multiplication)
- Now, is ϕ true in S ?

Satisfying structures

- Definition: A formula is satisfiable if there exists a structure that satisfies it

- Example: $\phi = \exists x.\ x + 0 = 1$ is satisfiable
- Consider the structure S':
  - Domain: $\mathcal{N}_0$
  - Interpretation:
    - '0' and '1' are mapped to 0 and 1 in $\mathcal{N}_0$
    - '=' $\mapsto$ = (equality)
    - '+' $\mapsto$ + (addition)
- S' satisfies ϕ. S' is said to be a model of ϕ.

First-order theories

- First-order logic is a framework.
- It gives us a generic syntax and building blocks for constructing restrictions thereof.
- Each such restriction is called a first-order theory.

- A theory defines
  - the signature Σ (the set of nonlogical symbols) and
  - the interpretations that we can give them.

Definitions

- Σ – the signature. This is a set of nonlogical symbols.
- Σ-formula: a formula over Σ symbols + logical symbols.
- A variable is free if it is not bound by a quantifier.
- A sentence is a formula without free variables.
- A Σ-theory T is defined by a set of Σ-sentences.

Definitions…

- Let T be a Σ-theory
- A Σ-formula ϕ is T-satisfiable if there exists a structure that satisfies both ϕ and the sentences defining T.
- A Σ-formula ϕ is T-valid if all structures that satisfy the sentences defining T also satisfy ϕ.

Example

- Let Σ = {0,1, '+', '='}
- Recall $\phi = \exists x.\ x + 0 = 1$
- ϕ is a Σ-formula.
- We now define the following Σ-theory:
  - $\forall x.\ x = x$ // '=' must be reflexive
  - $\forall x,y.\ x + y = y + x$ // '+' must be commutative

- Not enough to prove the validity of ϕ !

Theories through axioms

- The number of sentences that are necessary for defining a theory may be large or infinite.
- Instead, it is common to define a theory through a set of axioms.
- The theory is defined by these axioms and everything that can be inferred from them by a sound inference system.

Example 1

- Let Σ = {'='}
  - An example Σ-formula is $\phi = ((x = y) \land \neg (y = z)) \rightarrow \neg(x = z)$
- We would now like to define a Σ-theory T that will limit the interpretation of '=' to equality.
- We will do so with the equality axioms:
  - $\forall x.\ x = x$ (reflexivity)
  - $\forall x,y.\ x = y \rightarrow y = x$ (symmetry)
  - $\forall x,y,z.\ x = y \land y = z \rightarrow x = z$ (transitivity)
- Every structure that satisfies these axioms also satisfies ϕ above.
- Hence ϕ is T-valid.

Example 2

- Let Σ = {'<'}
- Consider the Σ-formula ϕ: $\forall x\ \exists y.\ y < x$
- Consider the theory T:
    - $\forall x,y,z.\ x < y \wedge y < z \rightarrow x < z$ (transitivity)
    - $\forall x,y.\ x < y \rightarrow \neg(y < x)$ (anti-symmetry)

Example 2 (cont'd)

- Recall: ϕ: ∀x ∃y. y < x
- Is ϕ T-satisfiable?
- We will show a model for it.
  - Domain: $\mathcal{Z}$
  - '<' $\mapsto$ <
- Is ϕ T-valid ?
- We will show a structure to the contrary
  - Domain: $\mathcal{N}_0$
  - '<' $\mapsto$ <

Fragments

- So far we only restricted the nonlogical symbols.
- Sometimes we want to restrict the grammar and the logical symbols that we can use as well.
- These are called logic fragments.
- Examples:
  - The quantifier-free fragment over Σ = {‘=’, ‘+’,0,1}
  - The conjunctive fragment over Σ = {‘=’, ‘+’,0,1}

Fragments

- Let $\Sigma = \{\}$
  - (T must be empty: no nonlogical symbols to interpret)
- Q: What is the quantifier-free fragment of T ?
- A: propositional logic
- Thus, propositional logic is also a first-order theory.
  - A very degenerate one.

Theories

- Let $\Sigma = \{\}$
  - (T must be empty: no nonlogical symbols to interpret)
- Q: What is T ?
- A: Quantified Boolean Formulas (QBF)

- Example:
  - $\forall x_1 \exists x_2 \forall x_3 .\ x_1 \rightarrow (x_2 \vee x_3)$

Some famous theories

- Presburger arithmetic: Σ = {0,1, ‘+’, ‘=’}
- Peano arithmetic: Σ = {0,1, ‘+’, ‘*’, ‘=’}
- Theory of reals
- Theory of integers
- Theory of arrays
- Theory of pointers
- Theory of sets
- Theory of recursive data structures
- …

The algorithmic point of view...

- It is also common to present theories NOT through the axioms that define them.
- The interpretation of symbols is fixed to their common use.
  - Thus ‘+’ is plus, ...
- The fragment is defined via grammar rules rather than restrictions on the generic first-order grammar.

The algorithmic point of view...

- Example: equality logic (= “the theory of equality”)
- *Grammar:*

formula : *formula* $\vee$ *formula* | $\neg$ *formula* | *atom*

atom : term-variable = term-variable
| term-variable = constant | Boolean-variable

- Interpretation:
  ‘=’ is equality.

The algorithmic point of view...

- This simpler way of presenting theories is all that is needed when our focus is on decision procedures specific for the given theory.

- The traditional way of presenting theories is useful when discussing generic methods (for any decidable theory T)
  - Example 1: algorithms for combining two or more theories
  - Example 2: generic SAT-based decision procedure given a decision procedure for the conjunctive fragment of T.

Expressiveness of a theory

- Each formula defines a language:
  the set of satisfying assignments ('models') are the words accepted by this language.

- Consider the fragment '2-CNF'

  formula : (*literal* $\vee$ *literal*) | *formula* $\wedge$ *formula*

  literal: Boolean-variable | $\neg$Boolean-variable

$$(x_1 \vee \neg x_2) \wedge (\neg x_3 \vee x_2)$$

Expressiveness of a theory

- Now consider a Propositional Logic formula

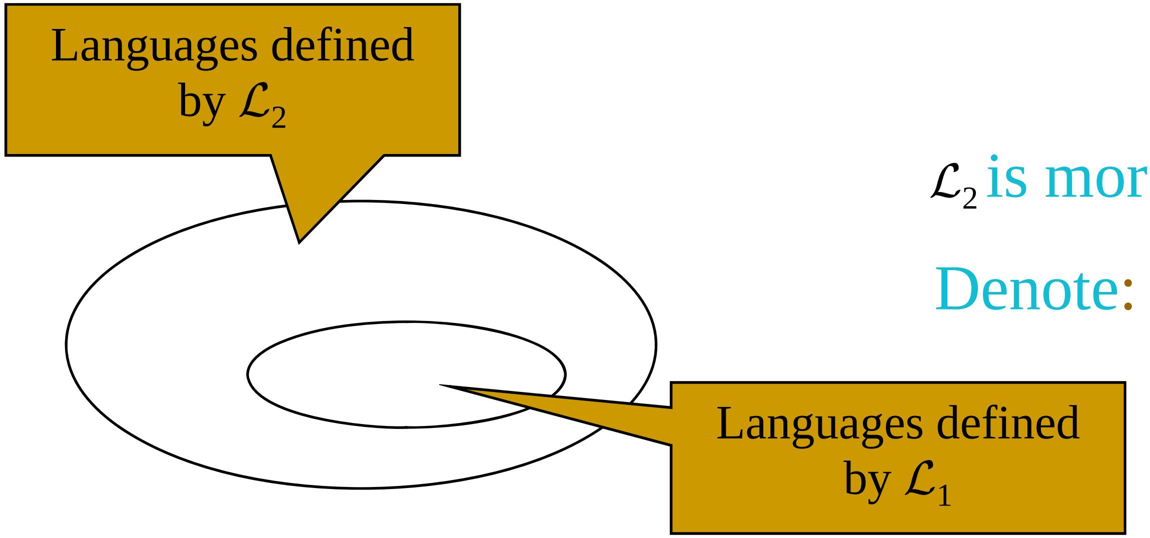
  ϕ: $(x_1 \vee x_2 \vee x_3)$.

- Q: Can we express this language with 2-CNF?
- A: No.

  Proof:

  - The language accepted by ϕ has 7 words: all assignments other than $x_1 = x_2 = x_3 = F$.
  - The first 2-CNF clause removes ¼ of the assignments, which leaves us with 6 accepted words. Additional clauses only remove more assignments.

Expressiveness of a theory



Languages defined by $\mathcal{L}_2$

Languages defined by $\mathcal{L}_1$

$\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$.

Denote: $\mathcal{L}_1 \prec \mathcal{L}_2$

- *Claim*: 2-CNF $\prec$ Propositional Logic
- Generally there is only a partial order between theories.

The tradeoff

- So we see that theories can have different expressive power.

- Q: why would we want to restrict ourselves to a theory or a fragment ? why not take some ‘maximal theory’…

- A: Adding axioms to the theory may make it harder to decide or even undecidable.

Example: Hilbert axiom system ($\mathcal{H}$)

- Let $\mathcal{H}$ be (M.P) + the following axiom schemas:

$$\frac{}{A \rightarrow (B \rightarrow A)} \quad (H1)$$

$$\frac{}{((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))} \quad (H2)$$

$$\frac{}{(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)} \quad (H3)$$

- $\mathcal{H}$ is sound and complete
- This means that with $\mathcal{H}$ we can prove any valid propositional formula, and only such formulas. The proof is finite.

Example

- But there exists first order theories defined by axioms which are not sufficient for proving all T-valid formulas.

Example: First Order Peano Arithmetic

- Σ = {0,1,‘+’, ‘*’, ‘=’}
- Domain: Natural numbers

Undecidable!

- Axioms (“semantics”):
  1. $\forall x : (0 \neq x + 1)$
  2. $\forall x : \forall y : (x \neq y) \rightarrow (x + 1 \neq y + 1)$
  3. Induction
  4. $\forall x : x + 0 = x$
  5. $\forall x : \forall y : (x + y) + 1 = x + (y + 1)$
  6. $\forall x : x * 0 = 0$
  7. $\forall x \forall y : x * (y + 1) = x * y + x$

+ (4., 5.)

* (6., 7.)

These axioms define the semantics of ‘+’

Example: First Order Presburger Arithmetic

- Σ = {0,1,‘+’, ~~‘*’~~, ‘=’}
- Domain: Natural numbers
- Axioms (“semantics”):

decidable!

1. $\forall x : (0 \neq x + 1)$
2. $\forall x : \forall y : (x \neq y) \rightarrow (x + 1 \neq y + 1)$
3. Induction

+

4. $\forall x : x + 0 = x$
5. $\forall x : \forall y : (x + y) + 1 = x + (y + 1)$

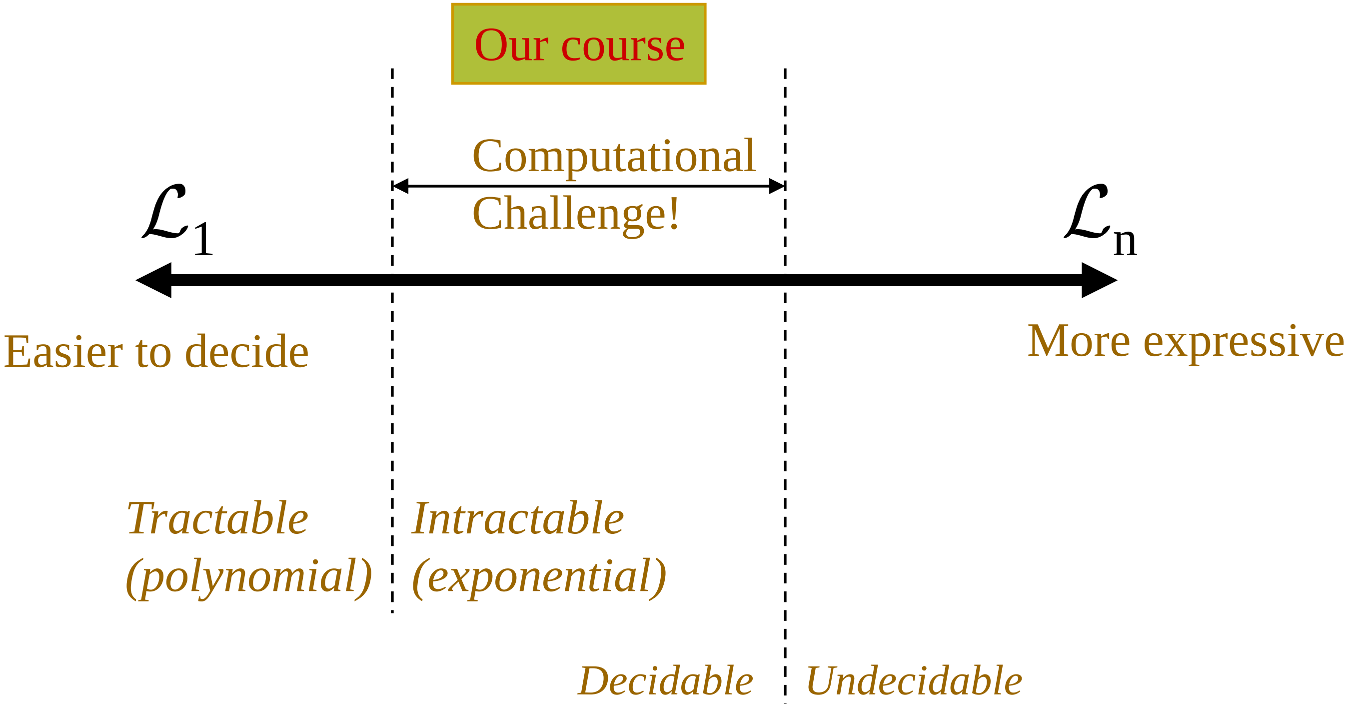
These axioms define the semantics of ‘+’

*

6. ~~$\forall x : x * 0 = 0$~~
7. ~~$\forall x \forall y : x * (y + 1) = x * y + x$~~

Tradeoff: expressiveness/computational hardness.

- Assume we are given theories $\mathcal{L}_1 \prec \ldots \prec \mathcal{L}_n$



Our course

Computational Challenge!

$\mathcal{L}_1$

$\mathcal{L}_n$

Easier to decide

More expressive

Tractable (polynomial)

Intractable (exponential)

Decidable

Undecidable

When is a specific theory useful?

1. Expressible enough to state something interesting.
2. Decidable (or semi-decidable) and more efficiently solvable than richer theories.
3. More expressible, or more natural for expressing some models in comparison to ‘leaner’ theories.

Expressiveness and complexity

- Q1: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two theories whose satisfiability problem is decidable and in the same complexity class. Is the satisfiability problem of an $\mathcal{L}_1$ formula reducible to a satisfiability problem of an $\mathcal{L}_2$ formula?

- Q2: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two theories whose satisfiability problems are reducible to one another. Are $\mathcal{L}_1$ and $\mathcal{L}_2$ in the same complexity class ?