Introduction to Logic (2/2)

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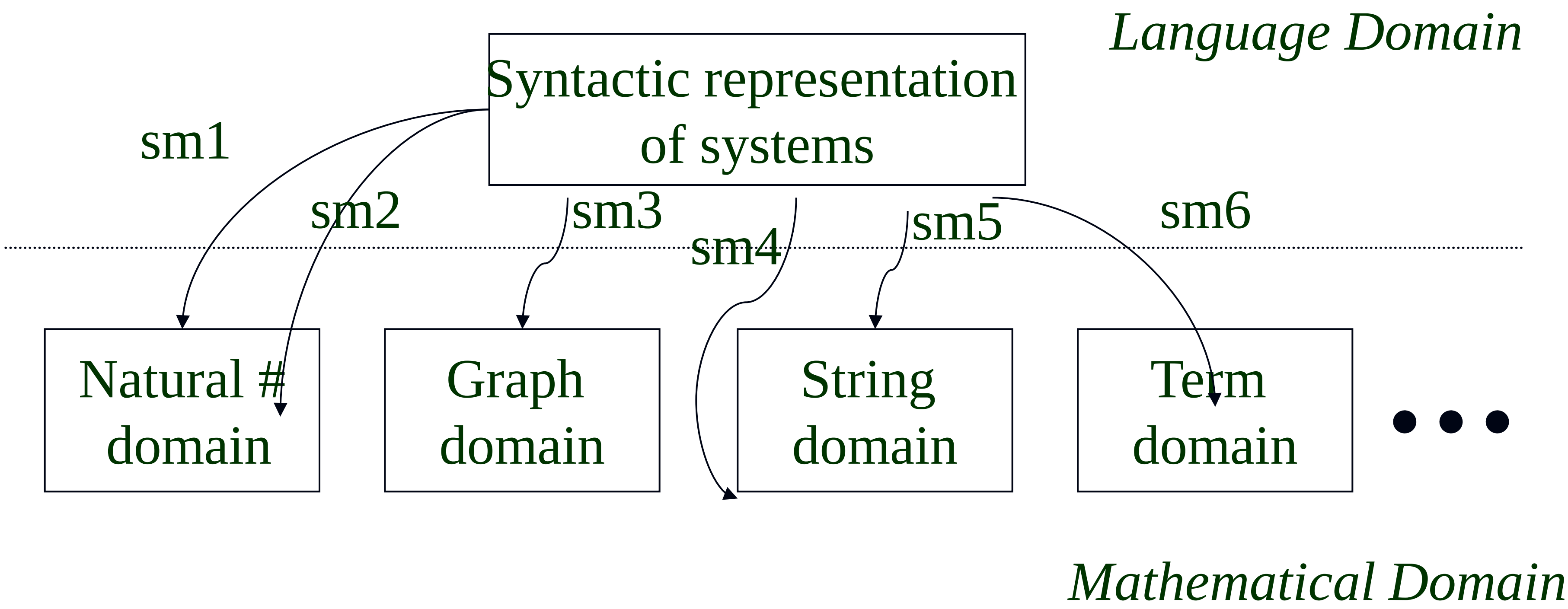
Overview

- Syntax v.s. semantics
- Various logics
- The history of mathematical logic
  - Logic at ancient Greek
  - Logic in 19th century
- Propositional calculus
  - Derivability/provability (symbolic manipulation)
  - Truth (semantic evaluation)
  - Soundness and completeness

Syntax v.s. Semantics

- An example of small language
  - BNF
    - F := 0 | 1 | F + 1 | 1 + F
    - Ex. 0, 0+1+1, 1+0+1, but not 0+0
  - Possible semantics
    - Is a formula 1 + 1 equal (*in what sense?*) to 1 + 1 + 0 ?
      - Yes (interpreting formula as natural number arithmetic),
        - $[1 + 1]_{N1} = 2$, $[1 + 1 + 0]_{N1} = 2$ $\rightarrow 1 + 1 =_{N1} 1 + 1 + 0$
      - No (interpreting formula as string),
        - $[1 + 1]_{S}$ = "1+1", $[1 + 1 + 0]_{S}$ = "1+1+0" $\rightarrow 1 + 1 \neq_{S} 1 + 1 + 0$
      - No (interpreting formula as natural # of string length)
        - $[1 + 1]_{N2} = 3$, $[1 + 1 + 0]_{N2} = 5$ $\rightarrow 1 + 1 =_{N2} 1 + 1 + 0$

Semantics Domain (cont.)



Various Logics 1/2

- Using a logic, we would like to specify requirement specification as a logical formula ϕ
- At the same time, we would like to prove whether a given system design M satisfies ϕ or not using an algorithm
- Therefore, we can characterize logic according to both
  - Expressive power
    - Ex. Second order logic > First order logic > Propositional logic
  - Computational complexity to prove $M \vdash \phi$ or $M \vDash \phi$
    - Ex. Propositional logic is decidable, i.e., every formula ϕ in the propositional logic can be proved mechanically
    - Ex. First order logic is undecidable, i.e., some formula ϕ in the first order logic cannot be proved using computer

Various Logics 2/2

- Suppose that the multiple readers/writers system has 10000 readers. Then, describing ϕ_{CON} as $(R_1 \wedge R_2) \vee (R_2 \wedge R_3) \vee (R_3 \wedge R_4) \ldots$ in propositional logic would have to write $(6 \times {}_{10000}C_2 - 1) = 3 \times 10^8$ characters.
    - For infinitely many readers, such way of description is even not possible.
- We can describe the requirement in the first order logic
    - $\exists i \; \exists j \; ((i \neq j) \wedge (R(i) \wedge R(j)))$ for some time instant t
- We can even describe the temporal condition in the requirement using the temporal logic
    - $\diamond \; \exists i \; \exists j \; ((i \neq j) \wedge (R(i) \wedge R(j)))$
    - More correctly, ϕ_{CON} should be $\Box \diamond \; \exists i \; \exists j \; ((i \neq j) \wedge (R(i) \wedge R(j)))$

Logic at Ancient Greek 1/2

- English word ‘trivial’ originates from
  - “trie” (3 = grammar, rhetoric, and logic) + “via” (way)
- The study of logic was begun by the ancient Greeks to formalize deduction
  - The derivation of true statements, called conclusions, from statements that are assumed to be true, called premises
  - Rhetoric (수사학) included the study of logic so that all sides in a debate would use the same *rules* of deduction
- *Axiom, theorem, and lemma* are ancient Greek words

Logic at Ancient Greek 2/2

- One such famous rule is the syllogism (삼단논법)
  - Premise1: All men are mortal
  - Premise2: X is a man
  - Conclusion: Therefore, X is mortal.
- Using the syllogism, we can deduce
  - Socrates is mortal
- However, careless use of logic can lead to claims that false statements are true or vice versa.
  - Premise1: Some cars make noise.
  - Premise2: My car is some car
  - Conclusion: Therefore, my car makes noise.

Logic at 19th Century

- Until the 19th century, logic remained a philosophical rather than a mathematical and scientific tool because
  - a natural language cannot express what mathematicians want to express and reason precisely enough
    - symbolic logic (a.k.a mathematical logic) was invented for the purpose in the 19th century where
      - formal symbols (e.g. "φ", "∧") are used to describe a formula instead of natural languages
        - Separation of a syntactic representation of a formula from its interpretation
      - formal rules to manipulate a formula purely based on its syntactic representation are defined

Logic at 19th Century

- 19th century, mathematicians questioned the legitimacy of the entire deductive process used to prove theorems in mathematics since they discovered the paradoxes
  - A Cretan philosopher says that "All Cretan people are liars" ???
  - The Sophist's Paradox.
    - A Sophist is sued for his tuition by the school that educated him.
      - He argues that he must win, since, if he loses, the school didn't educate him well enough, and doesn't deserve the money.
      - The school argues that he must lose, since, if he wins, he was educated well enough and therefore should pay for it.
  - Russell's paradox (1902)
    - Consider the set *A* of all those sets *X* such that *X* is not a member of *X*.
    - Clearly, by definition, *A* is a member of *A* if and only if *A* is not a member of *A*. So,
      - if *A* is a member of *A*, then *A* is also not a member of *A*
      - If *A* is not a member of *A*, then *A* is a member of *A*
    - In a formal way, consider the set $T=\{ S \mid S \notin S\}$
      - Then $T \in T \leftrightarrow T \notin T$ (a contradiction)

Logic at 19th Century

- Thus, they wanted to justify mathematical deduction by formalizing a system of logic in which the set of derivable/provable statements is the same as the set of true statements, i.e.,
  1. Every statement that can be proved is true
  2. If a statement is in fact true, there is a proof for the statement

An Example of a Provable Statement and a True Statement

Four derivation rules

$$\frac{\varphi \wedge \psi}{\varphi} \wedge E_l \qquad \frac{\varphi \wedge \psi}{\psi} \wedge E_r$$

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge I \qquad \frac{[\varphi] \quad \psi}{\varphi \rightarrow \psi} \rightarrow I$$

assumption

$$\cfrac{\cfrac{\cfrac{[\varphi \wedge \psi]}{\psi} \wedge E_r \qquad \cfrac{[\varphi \wedge \psi]}{\varphi} \wedge E_l}{\psi \wedge \varphi} \wedge I}{\varphi \wedge \psi \rightarrow \psi \wedge \varphi} \rightarrow I$$

Proof
- **Syntactic method**
- **Derivation**

$$\vdash \varphi \wedge \psi \rightarrow \psi \wedge \varphi$$

φ	ψ	φ ∧ ψ → ψ ∧ φ
T	T	T
T	F	T
F	T	T
F	F	T

Validity
- **Semantic method**
- **Valuation**

$$\models \varphi \wedge \psi \rightarrow \psi \wedge \varphi$$

Logic at 19th Century

- Hilbert's program, the research spurred by this plan, resulted in the development of systems of logic
  - Also, development of theories of the nature of logic itself
- Gödel showed that there are true statements of arithmetic that are not provable. This famous theorem is called *Gödel's incompleteness theorem*.
  - Thus, Gödel's incompleteness theorem refutes Hilbert's program's goal.
- The application of logic to computer science has spurred the development of new systems of logic
  - Analogy to cross-fertilization between continuous mathematics and applications in the physical sciences

Propositional Calculus

- The study of logic commences with the study of reasoning truth of sentences. Thus, sentential logic is the most primitive logic and also known as propositional logic.
  - A proposition *p* represents a declarative sentence.
    - A proposition *p* states that "John eats an apple"
    - A proposition *q* states that "Mary eats an orange"
- Formulas of the propositional logic are defined by syntactical rules using Boolean operators ($\neg, \rightarrow, \wedge, \vee$)
  - Suppose that φ and ψ are well-formed propositional formulas (wff). Every proposition is a well-formed formula. Then,
    - (φ), $\neg\, \varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ are wffs, too.
      - Note that $\neg$ and $\wedge$ are core Boolean operators
    - $((\rightarrow\psi$ is not a wff.

Propositional Calculus

- Syntax is also used to define the concept of proof, the symbolic manipulation of formulas in order to deduce a theorem.
  - See derivation rules and proof tree at slide 12
- Meaning (semantics) of the formula is defined by interpretations which assign a value true or false to every formula.
  - See truth table at slide 12
- Propositional logic is sound and complete in a sense that
  - Derivability coincide with truth
    - A wff φ can be proved if and only if φ is true
    - In other words, if you can prove φ using derivation rules, then φ must be evaluated true using the truth table. Also, vice versa.

Greek Letters

Name		Name		Name	
alpha	α	beta	β	Gamma	Γ
gamma	γ	delta	δ	Theta	Θ
epsilon	ε	zeta	ζ	Xi	Ξ
eta	η	theta	θ	Omega	Ω
iota	ι	kappa	κ	Pi	Π
lambda	λ	mu	μ	Delta	Δ
nu	ν	xi	ξ	Lambda	Λ
chi	χ	pi	π	Phi	Φ
rho	ρ	upsilon	υ	Sigma	Σ
phi	ϕ	psi	ψ	Psi	Ψ
omega	ω				