

# Predicate Calculus - Semantics 1/4

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# Introduction to predicate calculus (1/2)

- Propositional logic (sentence logic) dealt quite satisfactorily with sentences using conjunctive words (접속사) like **not**, **and**, **or**, and **if ... then**. But it fails to reflect the *finer logical structure* of the sentence
- What can we reason about a sentence itself which deals with its **target domain**?
  - ex. Jane is taller than Alice (target domain : human being)
  - ex2. For natural numbers  $x$  and  $y$ ,  $x+y \geq -(x+y)$  (target domain:  $\mathcal{N}$ )
- What can we reason about a sentence itself which also deal with modifiers like **there exists...**, **all ...**, **among ...** and **only ....** ?
  - Note that these modifiers enable us to reason about an **infinite** domain because we do not have to enumerate all elements in the domain

# Introduction to predicate calculus (2/2)

- Ex. Every student is younger than some instructor
  - We could simply identify this assertion with a propositional atom  $p$ .  
However, this fails to reflect the finer logical structure of this sentence
- This statement is about **being a student**, **being an instructor** and **being younger than somebody else** for a set of university members as a target domain
  - We need to express them and use **predicates** for this purpose
  - $S(\text{yunho})$ ,  $I(\text{moonzoo})$ ,  $Y(\text{yunho}, \text{moonzoo})$
- We need variables  $x$ ,  $y$  to not to write down all instance of  $S(-)$ ,  $I(-)$ ,  $Y(-)$ 
  - Every student  $x$  is young than some instructor  $y$
- Finally, we need **quantifiers** to capture the actual elements by variables
  - For every  $x$ , if  $x$  is a student, then there is some  $y$  which is an instructor such that  $x$  is younger than  $y$
  - $\forall x (S(x) \rightarrow (\exists y (I(y) \wedge Y(x,y))))$

# Relations and predicates

- The axioms and theorems of mathematics are defined on **arbitrary sets (domain)** such as the set of integers  $\mathbb{Z}$ 
  - ex. Fermat's last theorem
    - If an integer  $n$  is greater than 2, then the equation  $a^n + b^n = c^n$  has no solutions in non-zero integers  $a$ ,  $b$ , and  $c$ .
    - Can we express the Fermat's last theorem in propositional logic?
- The **predicate calculus** extends the propositional calculus with **predicate** letters that are interpreted as **relations on a domain**
  - i.e., predicates are interpreted upon domain
- Def 5.2. A relation can be represented by a boolean valued function  $R:D^n \rightarrow \{T,F\}$ , by mapping an  $n$ -tuple to T iff it is included in the relation
  - $R(d_1, \dots, d_n) = T$  iff  $(d_1, \dots, d_n) \in \mathcal{R}$

# Predicate formulas

- Let  $\mathcal{P}$ ,  $\mathcal{A}$  and  $\mathcal{V}$  be countable sets of symbols called **predicate letters**, **constants**, and **variables**, respectively.

- $\mathcal{P}=\{p,q,r\}$   $\mathcal{A}=\{a,b,c\}$ ,  $\mathcal{V}=\{x,y,z\}$

- Def 5.4 Atomic formulas and formulas

- atomic formula

- argument ::=  $x$  for any  $x \in \mathcal{V}$
- argument ::=  $a$  for any  $a \in \mathcal{A}$
- argument\_list ::= argument<sup>+</sup>
- atomic\_formula ::=  $p$  |  $p(\text{argument\_list})$  for any  $p \in \mathcal{P}$

- formula ::= atomic\_formula
- formula ::=  $\neg$  formula
- formula ::= formula  $\vee$  formula
- formula ::=  $\forall x$  formula
- formula ::=  $\exists x$  formula

- $\forall x \forall y (p(x, y) \rightarrow p(y, x)).$

- $\forall x \exists y p(x, y).$

- $\exists x \exists y (p(x) \wedge \neg p(y)).$

- $\forall x p(a, x).$

- $\forall x (p(x) \wedge q(x)) \leftrightarrow (\forall x p(x) \wedge \forall x q(x)).$

- $\exists x (p(x) \vee q(x)) \leftrightarrow (\exists x p(x) \vee \exists x q(x)).$

- $\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x)).$

- $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x)).$

# Free and bound variables

- Def 5.6
  - $\forall$  is the **universal quantifier** and is read ‘for all’.
  - $\exists$  is the **existential quantifier** and is read ‘there exists’.
  - In a quantified formula  $\forall xA$ ,  $x$  is the **quantified variable** and  $A$  is the **scope** of the quantified variable.
- Def 5.7 Let  $A$  be a formula. An occurrence of a variable  $x$  in  $A$  is a **free variable** of  $A$  iff  $x$  is not within the scope of a quantified variable  $x$ .
  - Notation:  $A(x_1, \dots, x_n)$  indicates that the set of free variables of the formula  $A$  is a subset of  $\{x_1, \dots, x_n\}$ . A variable which is not free is **bound**.
  - If a formula has no free variable it is **closed**
  - If  $\{x_1, \dots, x_n\}$  are all the free variables of  $A$ , the **universal closure** of  $A$  is  $\forall x_1 \dots \forall x_n A$  and the **existential closure** is  $\exists x_1 \dots \exists x_n A$
- Ex 5.8  $p(x,y), \exists y p(x,y), \forall x \exists y p(x,y)$
- Ex 5.9
  - In  $(\forall x p(x)) \wedge q(x)$ , the occurrence of  $x$  in  $p(x)$  is bound and the occurrence in  $q(x)$  is free. The universal closure is  $\forall x (\forall x p(x) \wedge q(x))$ .
  - Obviously, it would have been better to write the formula as  $\forall x p(x) \wedge q(y)$  where  $y$  is the free variable

# Interpretations (1/5)

- Def 5.10 Let  $U$  be a set of formulas s.t.  $\{p_1, \dots, p_m\}$  are all the predicate letters and  $\{a_1, \dots, a_k\}$  are all the constant symbols appearing in  $U$ . An **interpretation**  $\mathcal{I}$  is a triple  $(D, \{R_1, \dots, R_m\}, \{d_1, \dots, d_k\})$ , where
  - $D$  is a **non-empty** set,
  - $R_i$  is an  $n_i$ -ary relation on  $D$  that is assigned to the  $n_i$ -ary predicate  $p_i$ 
    - Notation:  $p_i^{\mathcal{I}} = R_i$
  - $d_i \in D$  is an element of  $D$  that is assigned to the constant  $a_i$ 
    - Notation:  $a_i^{\mathcal{I}} = d_i$
- Ex 5.11. Three numerical interpretations for  $\forall x p(a, x)$ :
  - $\mathcal{I}_1 = (\mathcal{N}, \{\leq\}, \{0\})$ ,  $\mathcal{I}_2 = (\mathcal{N}, \{\leq\}, \{1\})$ .  $\mathcal{I}_3 = (\mathcal{Z}, \{\leq\}, \{0\})$ .
  - $\mathcal{I}_4 = (\mathcal{S}, \{\text{substr}\}, \{\text{" "}\})$  where  $\mathcal{S}$  is the set of strings on some alphabet

# Interpretations (2/5)

- Def 5.12 Let  $\mathcal{I}$  be an interpretation. An assignment  $\sigma_{\mathcal{I}}: \mathcal{V} \rightarrow \mathbf{D}$  is a function which maps every variable to an element of the domain of  $\mathcal{I}$ .  $\sigma_{\mathcal{I}}[x_i \leftarrow d_i]$  is an assignment that is the same as  $\sigma_{\mathcal{I}}$  except that  $x_i$  is mapped to  $d_i$
- Def 5.13 Let  $A$  be a formula,  $\mathcal{I}$  an interpretation and  $\sigma_{\mathcal{I}}$  an assignment.  $v_{\sigma_{\mathcal{I}}}(A)$ , the **truth value** of  $A$  **under**  $\sigma_{\mathcal{I}}$  is defined by induction on the structure of  $A$ :
  - Let  $A = p_k(c_1, \dots, c_n)$  be an atomic formula where each  $c_i$  is either a variable  $x_i$  or a constant  $a_i$ .  $v_{\sigma_{\mathcal{I}}}(A) = \mathbf{T}$  iff
    - $\langle d_1, \dots, d_n \rangle \in R_k$  where  $R_k$  is the relation assigned by  $\mathcal{I}$  to  $p_k$  and
    - $d_i$  is the domain element assigned to  $c_i$ , either
      - by  $\mathcal{I}$  if  $c_i$  is a constant or
      - by  $\sigma_{\mathcal{I}}$  if  $c_i$  is variable
  - $v_{\sigma_{\mathcal{I}}}(\neg A) = \mathbf{T}$  iff  $v_{\sigma_{\mathcal{I}}}(A) = \mathbf{F}$
  - $v_{\sigma_{\mathcal{I}}}(A_1 \vee A_2) = \mathbf{T}$  iff  $v_{\sigma_{\mathcal{I}}}(A_1) = \mathbf{T}$  or  $v_{\sigma_{\mathcal{I}}}(A_2) = \mathbf{T}$
  - $v_{\sigma_{\mathcal{I}}}(\forall x A_1) = \mathbf{T}$  iff  $v_{\sigma_{\mathcal{I}}[x \leftarrow d]}(A_1) = \mathbf{T}$  for **all**  $d \in \mathbf{D}$
  - $v_{\sigma_{\mathcal{I}}}(\exists x A_1) = \mathbf{T}$  iff  $v_{\sigma_{\mathcal{I}}[x \leftarrow d]}(A_1) = \mathbf{T}$  for **some**  $d \in \mathbf{D}$





# Interpretations (3/5)

- Thm 5.14 Let  $A$  be a closed formula. Then  $v_{\sigma_{\mathcal{I}}}(A)$  does **not** depend on  $\sigma_{\mathcal{I}}$ . In such cases, we use simply  $v_{\mathcal{I}}(A)$  instead of  $v_{\sigma_{\mathcal{I}}}(A)$
- (important!) Thm 5.15 Let  $A' = A(x_1, \dots, x_n)$  be a non-closed formula and let  $\mathcal{I}$  be an interpretation. Then:
  - $v_{\sigma_{\mathcal{I}}}(A') = T$  for **some** assignment  $\sigma_{\mathcal{I}}$  iff  $v_{\mathcal{I}}(\exists x_1 \dots \exists x_n A') = T$
  - $v_{\sigma_{\mathcal{I}}}(A') = T$  for **all** assignment  $\sigma_{\mathcal{I}}$  iff  $v_{\mathcal{I}}(\forall x_1 \dots \forall x_n A') = T$
  - Thm 5.15 is important since we have many chances to add or remove quantified variables to and from formula during proofs.
- Def 5.16 A closed formula  $A$  is true in  $\mathcal{I}$  or  $\mathcal{I}$  is a model for  $A$ , if  $v_{\mathcal{I}}(A) = T$ .
  - Notation:  $\mathcal{I} \models A$
  - Note that we overload  $\models$  with usual logical consequence as in propositional logic
    - $\{A_1, A_2, A_3\} \models A$
- Def 5.18 A closed formula  $A$  is **satisfiable** if for **some** interpretation  $\mathcal{I}$ ,  $\mathcal{I} \models A$ .  $A$  is **valid** if for **all** interpretations  $\mathcal{I}$ ,  $\mathcal{I} \models A$ 
  - Notation:  $\models A$ .

# Interpretation (4/5)

- Ex 5.19  $\models (\forall x p(x)) \rightarrow p(a)$ 
  - Suppose that it is **not**. Then there must be an interpretation  $\mathcal{I} = (D, \{R\}, \{d\})$  such that  $v_{\mathcal{I}}(\forall x p(x)) = T$  and  $v_{\mathcal{I}}(p(a)) = F$
  - By Thm 5.15,  $v_{\sigma_{\mathcal{I}}}(p(x)) = T$  for all assignments  $\sigma_{\mathcal{I}}$ , in particular for the assignment  $\sigma'_{\mathcal{I}}$  that assigns  $d$  to  $x$  (i.e.  $v_{\sigma'_{\mathcal{I}}}(p(x)) = T$ ). But  $p(a)$  is closed, so  $v_{\sigma'_{\mathcal{I}}}(p(a)) = v_{\mathcal{I}}(p(a)) = F$ , a **contradiction**

**Example 5.20** Here is a semantic analysis of the formulas from Example 5.5:

- $\forall x \forall y (p(x, y) \rightarrow p(y, x))$ 

The formula is satisfiable in an interpretation where  $p$  is assigned a symmetric relation like  $=$ .
- $\forall x \exists y p(x, y)$ 

The formula is satisfiable in an interpretation where  $p$  is assigned a relation that is a total function, such as  $(x, y) \in R$  iff  $y = x + 1$  for  $x, y \in \mathcal{Z}$ .
- $\exists x \exists y (p(x) \wedge \neg p(y))$ 

This formula is satisfiable only in a domain with at least two elements.

# Interpretation (5/5)

- $\forall x p(a, x)$

This expresses the existence of a special element. For example, if  $p$  is interpreted by the relation  $\leq$  on the domain  $\mathcal{N}$ , then the formula is true for  $a = 0$ . If we change the domain to  $\mathcal{Z}$  the formula is false for the same assignment of  $\leq$  to  $p$ . Thus a change of domain alone can falsify a formula.

- $\forall x(p(x) \wedge q(x)) \leftrightarrow (\forall x p(x) \wedge \forall x q(x))$

The formula is valid. We prove the forward direction and leave the converse as an exercise. Let  $\mathcal{I} = (D, \{R_1, R_2\}, \{ \})$  be an arbitrary interpretation. By Theorem 5.15,  $v_{\sigma_{\mathcal{I}}}(p(x) \wedge q(x)) = T$  for all all assignments  $\sigma_{\mathcal{I}}$ , and by the inductive definition of an interpretation,  $v_{\sigma_{\mathcal{I}}}(p(x)) = T$  and  $v_{\sigma_{\mathcal{I}}}(q(x)) = T$  for all assignments  $\sigma_{\mathcal{I}}$ . Again by Theorem 5.15,  $v_{\mathcal{I}}(\forall x p(x)) = T$  and  $v_{\mathcal{I}}(\forall x q(x)) = T$ , and by the definition of interpretation  $v_{\mathcal{I}}(\forall x p(x) \wedge \forall x q(x)) = T$ .

Show that  $\forall$  does not distribute over disjunction by constructing a falsifying interpretation for  $\forall x(p(x) \vee q(x)) \leftrightarrow (\forall x p(x) \vee \forall x q(x))$ .

- $\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$

This is a valid formula, but its converse is not.