Propositional Calculus
- Natural deduction

Moonzoo Kim
CS Dept. KAIST

moonzoo@cs.kaist.ac.kr

Sound & Complete

$\models \phi$ v.s. $\vdash \phi$

Truth	**Tool (natural deduction, $\mathcal{G}$, $\mathcal{H}$, etc)**
Semantics	**Syntax**
Validity	**Proof**
All interpretations	**Finite proof tree**
Undecidable (except prop. Logic)	**Manual heuristics**

Natural deduction

- In natural deduction, similar to other deductive proof systems such as $\mathcal{G}$ and $\mathcal{H}$, we have a collection of proof rules.
  - Natural deduction does not have axioms.
- Suppose we have premises .and $\phi_1, \phi_2, \ldots, \phi_n$ and would like to prove a conclusion ψ. The intention is denoted by

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

  We call this expression a sequent; it is valid if a proof for it can be found
- Def: A logical formula ϕ with valid sequent $\vdash \phi$ is theorem

Proof rules (1/3)

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$

- $\wedge$i says: to prove $\phi \wedge \psi$, you must first prove ϕ and ψ separately and then use the rule $\wedge$ i.
- $\wedge\ e_1$ says: to prove ϕ, try proving $\phi \wedge \psi$ and then use the rule $\wedge\ e_1$. Actually this does not sound like very good advice because probably proving $\phi \wedge \psi$ will be harder than proving ϕ alone. However, you might find that you already have $\phi \wedge \psi$ lying around, so that's when this rule is useful.

Proof rules (1/3)

	introduction	elimination
$\lor$	$\dfrac{\phi}{\phi \lor \psi}\ \lor i_1 \qquad \dfrac{\psi}{\phi \lor \psi}\ \lor i_2$	assumption 1 ... assumption 2 $\dfrac{\phi \lor \psi \quad \boxed{\begin{matrix}\boxed{\phi} \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\boxed{\psi} \\ \vdots \\ \chi\end{matrix}}}{\chi}\ \lor e$

- $\lor$ i_1 says: to prove $\phi \lor \psi$, try proving ϕ. Again, in general it is harder to prove ϕ than it is to prove $\phi \lor \psi$, so this will usually be useful only if you have already managed to prove ϕ.
- $\lor$ e has an excellent procedural interpretation. It says: if you have $\phi \lor \psi$, and you want to prove some χ, then try to prove χ from ϕ and from ψ in turn
  - In those subproofs, of course you can use the other prevailing premises as well

Proof rules (3/3)

	introduction	*elimination*
$\to$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}\ {\to}\mathrm{i}$	$\dfrac{\phi \quad \phi \to \psi}{\psi}\ {\to}\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

Some useful derived rules

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad\qquad \frac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

- At any stage of a proof, it is permitted to introduce any formula as assumption, by choosing a proof rule that opens a box. As we saw, natural deduction employs boxes to control the scope of assumptions.
- When an assumption is introduced, a box is opened. Discharging assumptions is achieved by closing a box according to the pattern of its particular proof rule.

Example 1

$$p \land \neg q \to r,\ \neg r,\ p \vdash q$$

1	$p \land \neg q \to r$	premise
2	$\neg r$	premise
3	p	premise
4	$\neg q$	assumption
5	$p \land \neg q$	$\land$i 3, 4
6	r	$\to$e 1, 5
7	$\bot$	$\neg$e 6, 2
8	$\neg\neg q$	$\neg$i 4–7
9	q	$\neg\neg$e 8

Example 2

$$p \to q \vdash \neg p \lor q$$

1	$p \to q$	premise
2	$\neg p \lor p$	LEM
3	$\neg p$	assumption
4	$\neg p \lor q$	$\lor i_1$ 3
5	p	assumption
6	q	$\to$e 1, 5
7	$\neg p \lor q$	$\lor i_2$ 6
8	$\neg p \lor q$	$\lor$e 2, 3–4, 5–7

Example 3 (Law of Excluded Middle)

$$\frac{}{\phi \lor \neg\phi}\ \text{LEM}$$

1	$\neg(\phi \lor \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \lor \neg\phi$	$\lor i_1$ 2
4	$\bot$	$\neg e$ 3, 1
5	$\neg\phi$	$\neg i$ 2–4
6	$\phi \lor \neg\phi$	$\lor i_2$ 5
7	$\bot$	$\neg e$ 6, 1
8	$\neg\neg(\phi \lor \neg\phi)$	$\neg i$ 1–7
9	$\phi \lor \neg\phi$	$\neg\neg e$ 8

Summary of proof rules

	introduction	*elimination*
$\land$	$\dfrac{\phi \quad \psi}{\phi \land \psi}\ \land\mathrm{i}$	$\dfrac{\phi \land \psi}{\phi}\ \land\mathrm{e}_1 \qquad \dfrac{\phi \land \psi}{\psi}\ \land\mathrm{e}_2$
$\lor$	$\dfrac{\phi}{\phi \lor \psi}\ \lor\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \lor \psi}\ \lor\mathrm{i}_2$	$\dfrac{\phi \lor \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \lor\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

$$\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad \dfrac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad \dfrac{}{\phi \lor \neg\phi}\ \mathrm{LEM}$$

Proof Tips

- First, write down premises at the top of the paper
- Second, write down a conclusion at the bottom of the paper
- Third, look at the structure of the conclusion and try to find compatible proof rules backwardly
  - Pattern matching works, although not all the time.