Propositional Calculus
- *Gentzen System $\mathcal{G}$*

Moonzoo Kim
CS Dept. KAIST

moonzoo@cs.kaist.ac.kr

Review

- Goal of logic
  - To check whether given a formula ϕ is valid
  - To prove a given formula ϕ

Sound & Complete

$\vdash \phi$ $\equiv$ $\models \phi$

Syntactic method (proof system $\mathcal{G}$, $\mathcal{H}$, natural deduction,etc)

Semantic method (truth table, etc)

Sound & Complete $\equiv$ **Semantic tableau** $\equiv$ ***Sound & Complete***

Deductive proofs (1/3)

- Suppose we want to know if ϕ belongs to the theory $\mathcal{T}$(U).
  - By Thm 2.38 U $\vDash \phi$ iff $\vDash A_1 \wedge \ldots \wedge A_n \rightarrow \phi$ where U = { $A_1,\ldots,A_n$}
  - Thus, $\phi \in \mathcal{T}$(U) iff a decision procedure for validity answers 'yes'
- However, there are several problems with this semantic approach
  - The set of axioms may be infinite
    - e.x. Hilbert deductive system $\mathcal{H}$ has an axiom schema $(A \rightarrow (B \rightarrow A))$, which generates an infinite number of axioms by replacing schemata variables A,B and C with infintely many subformulas (e.g. $\phi\wedge\psi$,$\neg\phi\vee\psi$, etc)
    - e.x.2. Peano and ZFC theories cannot be finitely axiomatized.
  - Very few logics have decision procedures for validity of ϕ
    - ex. propositional logic has a decision procedure using truth table
    - ex2. predicate logic does not have such decision procedure
- There is another approach to logic called deductive proofs.
  - Instead of working with semantic concepts like interpretation/model and consequence
  - we choose a set of axioms and a set of syntactical rules for deducing new formulas from the axioms

Deductive proofs (2/3)

- Def 3.1
  - A deductive system consists of
    - a set of axioms and
    - a set of inference rules
  - A proof in a deductive system is a sequence of sets of formulas s.t. each element is either an axiom or it can be inferred from previous elements of the sequence using a rule of inference
  - If {A} is the last element of the sequence, A is a theorem, the sequence is a proof of A, and A is provable, denoted $\vdash$ A
- Example of a proof of $(p\vee q)\rightarrow(q\vee p)$ in gentzen system $\mathcal{G}$
  - $\{\neg p,q,p\}.\{\neg q,q,p\}.\{\neg(p\vee q),q,p\}.\{\neg(p\vee q),(q\vee p)\}.\{(p\vee q)\rightarrow(q\vee p)\}$

axioms

theorem

    - tree representation of this proof is more intuitive

$\neg p, q, p$ $\quad$ $\neg q, q, p$

$\searrow \; \beta\vee \; \swarrow$

$\neg(p \vee q), q, p$

$\downarrow \alpha\vee$

$\neg(p \vee q), (q \vee p)$

$\downarrow \alpha\rightarrow$

$(p \vee q) \rightarrow (q \vee p)$

Deductive proofs (3/3)

- Deductive proofs has following benefits
  - There may be an infinite number of axioms, but only a finite number of axioms will appear in any proof
  - Any particular proof consists of a finite sequence of sets of formulas, and the legality of each individual deduction can be easily and efficiently determined from the syntax of the formulas
  - The proof of a formula clearly shows which axioms, theorems and rules are used and for what purposes.
    - Such a pattern (i.e. relationship between formulas) can then be transferred to other similar proofs, or modified to prove different results.
    - Lemmas and corollaries can be obtained and can be used later
- But with a new problem
  - deduction is defined purely in terms of syntactical formula manipulation
  - But it is not amenable to systematic search procedures
    - no brute-force search is possible because any axiom can be used

The Gentzen system $\mathcal{G}$

- Def 3.2 The Gentzen system $\mathcal{G}$ is a deductive system.
  - The axioms are the sets of formulas containing a complementary pairs of literals
    - ex. { ¬p, p, p∧q} can be an axiom, but { ¬p, q, p∧q} is not.
  - The rules of inferences are:
    - note that a set of formulas in $\mathcal{G}$ is an implicit disjunction

premise

conclusion

$$\frac{\vdash U_1 \cup \{\alpha_1, \alpha_2\}}{\vdash U_1 \cup \{\alpha\}} \qquad \frac{\vdash U_1 \cup \{\beta_1\} \quad \vdash U_2 \cup \{\beta_2\}}{\vdash U_1 \cup U_2 \cup \{\beta\}}$$

8 α-rules

α	α_1	α_2
$\neg\neg A$	A	
$\neg(A_1 \wedge A_2)$	$\neg A_1$	$\neg A_2$
$A_1 \vee A_2$	A_1	A_2
$A_1 \rightarrow A_2$	$\neg A_1$	A_2
$A_1 \uparrow A_2$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \downarrow A_2)$	A_1	A_2
$\neg(A_1 \leftrightarrow A_2)$	$\neg(A_1 \rightarrow A_2)$	$\neg(A_2 \rightarrow A_1)$
$A_1 \oplus A_2$	$\neg(A_1 \rightarrow A_2)$	$\neg(A_2 \rightarrow A_1)$

7 β-rules

β	β_1	β_2
$B_1 \wedge B_2$	B_1	B_2
$\neg(B_1 \vee B_2)$	$\neg B_1$	$\neg B_2$
$\neg(B_1 \rightarrow B_2)$	B_1	$\neg B_2$
$\neg(B_1 \uparrow B_2)$	B_1	B_2
$B_1 \downarrow B_2$	$\neg B_1$	$\neg B_2$
$B_1 \leftrightarrow B_2$	$B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$
$\neg(B_1 \oplus B_2)$	$B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$

Soundness and completeness of $\mathcal{G}$

- Note that there are close relationship between a deductive proof of ϕ and semantic tableau of ϕ

$$\neg p, q, p \qquad\qquad \neg q, q, p$$
$$\searrow \qquad \swarrow$$
$$\neg(p \vee q), q, p$$
$$\downarrow$$
$$\neg(p \vee q), (q \vee p)$$
$$\downarrow$$
$$(p \vee q) \rightarrow (q \vee p)$$

A proof in $\mathcal{G}$

$$\neg[(p \vee q) \rightarrow (q \vee p)]$$
$$\downarrow$$
$$p \vee q, \neg(q \vee p)$$
$$\downarrow$$
$$p \vee q, \neg q, \neg p$$
$$\swarrow \qquad \searrow$$
$$p, \neg q, \neg p \qquad\qquad q, \neg q, \neg p$$
$$\times \qquad\qquad\qquad \times$$

Semantic tableau

Soundness and completeness of $\mathcal{G}$

- Thm 3.6 Let U be a set of formulas and $\bar{U}$ be the set of complements of formulas in U. Then $\vdash$U in $\mathcal{G}$ iff there is a closed semantic tableau T for $\bar{U}$
- Proof of completeness,
  - $\vdash$U in $\mathcal{G}$ if there exists a closed T for $\bar{U}$ exists
  - induction on the height of T, h
  - h=0
    - T consists of a single node labeled by $\bar{U}$, a set of literals containing a complementary pair (say $\{p, \neg p\}$), that is $\bar{U} = \bar{U}_0 \cup \{p, \neg p\}$
    - Obviously $U = U_0 \cup \{\neg p, p\}$ is an axiom in $\mathcal{G}$, hence $\vdash U$

Soundness and completeness of $\mathcal{G}$

- Proof of completeness (continued)
  - $\vdash U$ in $\mathcal{G}$ if there exists a closed T for $\bar{U}$ exists
  - h>0
    - Some tableau α or β rule was used at the root n of T on a formula $\bar{A} \in \bar{U}$, that is $\bar{U} = \bar{U}_0 \cup \{\bar{A}\}$
    - Case of α rule
      - A tableau α-rule was used on (a formula such as) $\bar{A} = \neg (A_1 \vee A_2)$ to produce the node n' labeled $\bar{U}' = \bar{U}_0' \cup \{ \neg A_1, \neg A_2\}$. The subtree rooted at n' is a closed tableau for $\bar{U}'$, so by the inductive hypothesis, $\vdash U_0 \cup \{A_1, A_2\}$. Using the α-rule in $\mathcal{G}$, $\vdash U_0 \cup \{A_1 \vee A_2\}$, that is $\vdash U$
    - Case of β rule
      - A tableau β-rule was used on (a formula such as) $\bar{A} = \neg (A_1 \wedge A_2)$ to produce the node n' and n" labeled $\bar{U}' = \bar{U}_0 \cup \{ \neg A_1\}$, $\bar{U}'' = \bar{U}_0 \cup \{\neg A_2\}$, respectively. By the inductive hypothesis, $\vdash U_0 \cup \{A_1\}$ and $\vdash U_0 \cup \{A_2\}$. Using the β-rule in $\mathcal{G}$, $\vdash U_0 \cup \{A_1 \wedge A_2\}$, that is $\vdash U$