Propositional Calculus - Gentzen System G

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Review

- Goal of logic
 - lacksquare To check whether given a formula ϕ is valid
 - lacksquare To prove a given formula ϕ



Deductive proofs (1/3)

- Suppose we want to know if ϕ belongs to the theory $\mathcal{T}(\mathsf{U})$.
 - By Thm 2.38 U $\models \phi$ iff $\models A_1 \land ... \land A_n \rightarrow \phi$ where U = { $A_1,...,A_n$ }
 - Thus, $\phi \in \mathcal{T}(U)$ iff a decision procedure for validity answers 'yes'
- However, there are several problems with this semantic approach
 - The set of axioms may be infinite
 - e.x. Hilbert deductive system \mathcal{H} has an axiom schema (A \rightarrow (B \rightarrow A)), which generates an infinite number of axioms by replacing schemata variables A,B and C with infinitely many subformulas (e.g. $\phi \land \psi, \neg \phi \lor \psi$, etc)
 - e.x.2. Peano and ZFC theories cannot be finitely axiomatized.
 - Very few logics have decision procedures for validity of ϕ
 - ex. propositional logic has a decision procedure using truth table
 - ex2. predicate logic does not have such decision procedure
- There is another approach to logic called deductive proofs.
 - Instead of working with semantic concepts like interpretation/model and consequence
 - we choose a set of axioms and a set of syntactical rules for deducing new formulas from the axioms



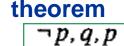
Def 3.1

Deductive proofs (2/3)

- A deductive system consists of
 - a set of axioms and
 - a set of inference rules
- A proof in a deductive system is a sequence of sets of formulas s.t. each element is either an axiom or it can be inferred from previous elements of the sequence using a rule of inference
- If {A} is the last element of the sequence, A is a theorem, the sequence is a proof of A, and A is provable, denoted ⊢ A
- Example of a proof of $(p \lor q) \rightarrow (q \lor p)$ in gentzen system \mathcal{G}

axioms

tree representation of this proof is more intuitive



$$\beta \lor \qquad \qquad \neg q, q, p$$

$$\neg (p \lor q), q, p$$

$$p \lor q), q, p$$

$$\bot \alpha \lor$$

$$\neg (p \lor q), (q \lor p)$$

$$\downarrow \alpha -$$

 $(p \lor q) \to (q \lor p)^4$

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Deductive proofs (3/3)

- Deductive proofs has following benefits
 - There may be an infinite number of axioms, but only a finite number of axioms will appear in any proof
 - Any particular proof consists of a finite sequence of sets of formulas, and the legality of each individual deduction can be easily and efficiently determined from the syntax of the formulas
 - The proof of a formula clearly shows which axioms, theorems and rules are used and for what purposes.
 - Such a pattern (i.e. relationship between formulas) can then be transferred to other similar proofs, or modified to prove different results.
 - Lemmas and corollaries can be obtained and can be used later
- But with a new problem
 - deduction is defined purely in terms of syntactical formula manipulation
 - But it is not amenable to systematic search procedures
 - no brute-force search is possible because any axiom can be used





The Gentzen system \mathcal{G}

- Def 3.2 The Gentzen system \mathcal{G} is a deductive system.
 - The axioms are the sets of formulas containing a complementary pairs of literals
 - ex. { $\neg p$, p, p $\land q$ } can be an axiom, but { $\neg p$, q, p $\land q$ } is not.
 - The rules of inferences are:
 - note that a set of formulas in \mathcal{G} is an implicit disjunction

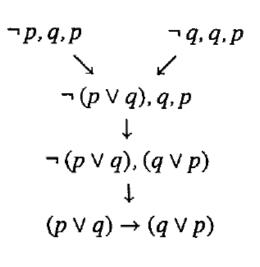
$$\frac{\vdash U_1 \cup \{\beta_1\} \quad \vdash U_2 \cup \{\beta_2\}}{\vdash U_1 \cup U_2 \cup \{\beta\}}$$

R

μ	p_1	p_2	
$B_1 \wedge B_2$	B_1	B ₂	
$\neg (B_1 \lor B_2)$	$\neg B_1$	¬ B ₂	β -rules
$\neg (B_1 \rightarrow B_1)$	B_2) B_1	$\neg B_2$	
$\neg (B_1 \uparrow B$	B_1	<i>B</i> ₂	\
$B_1 \downarrow B_2$	¬ B ₁	¬ B ₂	
$B_1 \leftrightarrow B_2$	$B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$	
$\neg (B_1 \oplus B$	$2) B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$	6

Soundness and completeness of \mathcal{G}

Note that there are close relationship between a deductive proof of ϕ and semantic tableau of ϕ



A proof in
$$\mathcal{G}$$

$$\neg[(p \lor q) \to (q \lor p)]
\downarrow
p \lor q, \neg (q \lor p)
\downarrow
p \lor q, \neg q, \neg p
\swarrow
p, \neg q, \neg p
x
q, \neg q, \neg p
x
x$$

Semantic tableau

Soundness and completeness of \mathcal{G}

- Thm 3.6 Let U be a set of formulas and Ū be the set of complements of formulas in U. Then ⊢U in G iff there is a closed semantic tableau T for Ū
- Proof of completeness,
 - \blacksquare \vdash U in \mathcal{G} if there exists a closed T for $\bar{\mathbf{U}}$ exists
 - induction on the height of T, h
 - h=0
 - T consists of a single node labeled by Ū, a set of literals containing a complementary pair (say {p, ¬p}), that is Ū = Ū₀ ∪ {p, ¬p}
 - Obviously $U = U_0 \cup \{\neg p, p\}$ is an axiom in \mathcal{G} , hence $\vdash U$



Soundness and completeness of \mathcal{G}

- Proof of completeness (continued)
 - \blacksquare \vdash U in \mathcal{G} if there exists a closed T for $\bar{\mathbf{U}}$ exists
 - h>0
 - Some tableau α or β rule was used at the root n of T on a formula $\bar{A} \in \bar{U}$, that is $\bar{U} = \bar{U}_0 \cup \{\bar{A}\}$
 - \blacksquare Case of α rule
 - A tableau α-rule was used on (a formula such as) Ā = ¬ (A₁ ∨ A₂) to produce the node n' labeled Ū' = Ū₀' ∪ { ¬A₁, ¬A₂}. The subtree rooted at n' is a closed tableau for Ū', so by the inductive hypothesis, ⊢ U₀ ∪ {A₁, A₂}. Using the α-rule in 𝒪, ⊢ U₀ ∪ {A₁ ∨ A₂}, that is ⊢ U
 - Case of β rule
 - A tableau β-rule was used on (a formula such as) Ā = ¬ (A₁ ∧ A₂) to produce the node n' and n" labeled Ū' = Ū₀ ∪ { ¬A₁}, Ū" = Ū₀ ∪ {¬A₂}, respectively. By the inductive hypothesis, ⊢ U₀ ∪ {A₁} and ⊢ U₀ ∪ {A₂}.

Using the β -rule in \mathcal{G} , $\vdash U_0 \cup \{A_1 \land A_2\}$, that is $\vdash U$



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