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# Introduction to Artificial Intelligence

## First-order Logic

(Logic, Deduction, Knowledge Representation)

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**UNIVERSITÄT KOBLENZ-LANDAU**

**Wintersemester 2003/2004**

# Outline

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- **Why first-order logic?**
- **Syntax and semantics of first-order logic**
- **Fun with sentences**
- **Wumpus world in first-order logic**

# Pros and Cons of Propositional Logic

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- 😊 Meaning in propositional logic is **context-independent**  
(unlike natural language, where meaning depends on context)
- 😞 Propositional logic has very limited expressive power  
(unlike natural language)

## Example:

Cannot say “pits cause breezes in adjacent squares”  
except by writing one sentence for each square

# First-order Logic

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## Propositional logic

Assumes that the world contains **facts**



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Assumes that the world contains

- **Objects**  
people, houses, numbers, theories, Donald Duck, colors, centuries, ...

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red, round, prime, multistoried, ...

brother of, bigger than, part of, has color, occurred after, owns, ...

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## Propositional logic

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- **Relations**  
red, round, prime, multistoried, ...  
brother of, bigger than, part of, has color, occurred after, owns, ...
- **Functions**  
+, middle of, father of, one more than, beginning of, ...

# Syntax of First-order Logic: Basic Elements

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## Symbols

**Constants**     *KingJohn, 2, Koblenz, C, ...*

**Predicates**     *Brother, >, =, ...*

**Functions**     *Sqrt, LeftLegOf, ...*

**Variables**     *x, y, a, b, ...*

**Connectives**      $\wedge \vee \neg \Rightarrow \Leftrightarrow$

**Quantifiers**      $\forall \exists$

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**Quantifiers**      $\forall$     $\exists$

## Note

The **equality predicate** is always in the vocabulary  
It is written in infix notation

# Syntax of First-order Logic: Atomic Sentences

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## Atomic sentence

$predicate ( term_1, \dots, term_n )$

or

$term_1 = term_2$

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## Atomic sentence

$predicate ( term_1, \dots, term_n )$

or

$term_1 = term_2$

## Term

$function ( term_1, \dots, term_n )$

or

$constant$

or

$variable$

# Syntax of First-order Logic: Atomic Sentences

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## Example

*Brother ( KingJohn, RichardTheLionheart )*




# Syntax of First-order Logic: Atomic Sentences

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*Brother* ( *KingJohn*, *RichardTheLionheart* )



# Syntax of First-order Logic: Atomic Sentences

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## Example

*Brother* ( *KingJohn*, *RichardTheLionheart* )

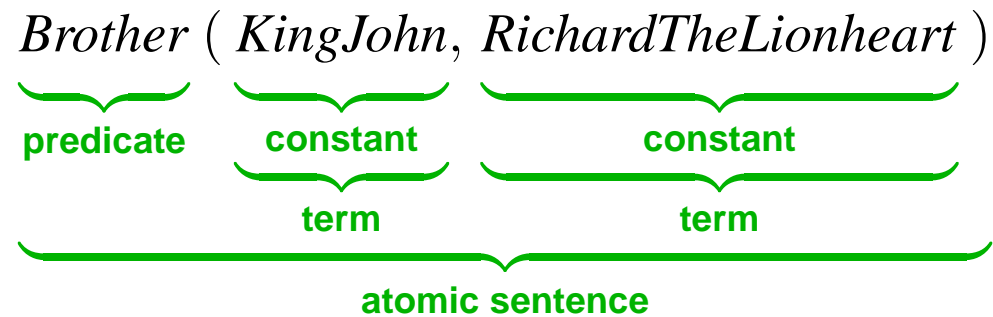
predicate      constant      constant

term      term

# Syntax of First-order Logic: Atomic Sentences

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## Example



# Syntax of First-order Logic: Atomic Sentences

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## Example

> (*Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn))*)

# Syntax of First-order Logic: Atomic Sentences

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## Example

$>$  ( $Length(LeftLegOf(Richard))$ ,  $Length(LeftLegOf(KingJohn))$ )

$\underbrace{\hspace{1.5em}}$   $\underbrace{\hspace{1.5em}}$   $\underbrace{\hspace{1.5em}}$   $\underbrace{\hspace{1.5em}}$   $\underbrace{\hspace{1.5em}}$   $\underbrace{\hspace{1.5em}}$   $\underbrace{\hspace{1.5em}}$

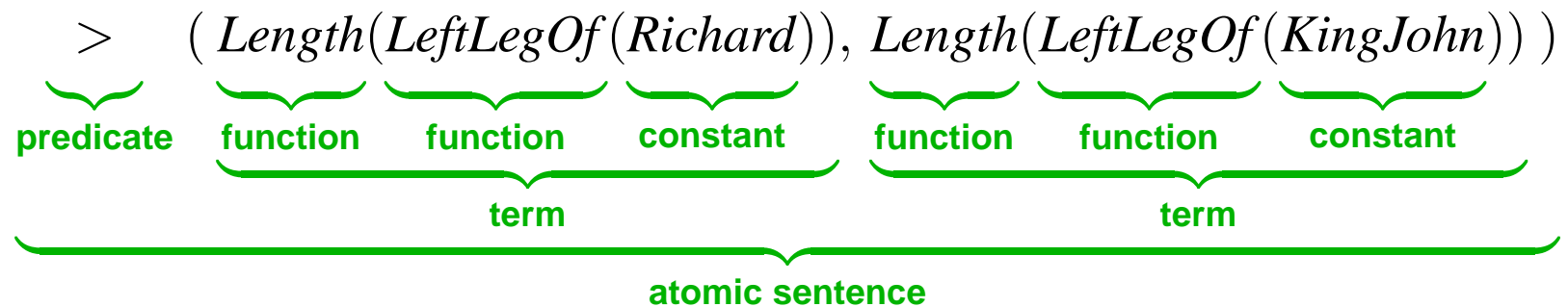
predicate    function    function    constant    function    function    constant



# Syntax of First-order Logic: Atomic Sentences

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## Example



# Syntax of First-order Logic: Complex Sentences

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Built from atomic sentences using connectives

$$\neg S \quad S_1 \wedge S_2 \quad S_1 \vee S_2 \quad S_1 \Rightarrow S_2 \quad S_1 \Leftrightarrow S_2$$

(as in propositional logic)



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Example

$$\textit{Sibling}(\textit{KingJohn}, \textit{Richard}) \Rightarrow \textit{Sibling}(\textit{Richard}, \textit{KingJohn})$$

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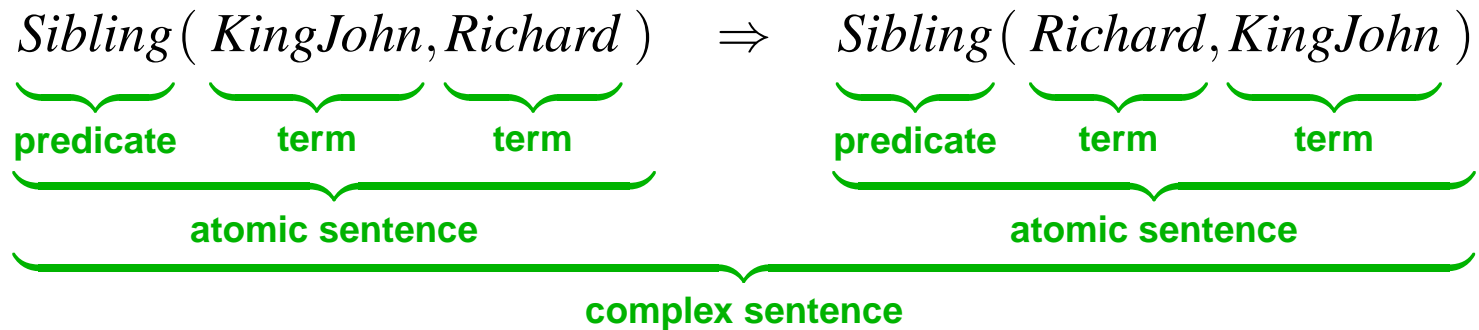
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# Semantics in First-order Logic

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## Models of first-order logic

Sentences are true or false with respect to models, which consist of

- a **domain** (also called universe)
- an **interpretation**

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## Domain

A non-empty (finite or infinite) set of arbitrary elements

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- an **interpretation**

## Domain

A non-empty (finite or infinite) set of arbitrary elements

## Interpretation

Assigns to each

- constant symbol: a domain element
- predicate symbol: a relation on the domain (of appropriate arity)
- function symbol: a function on the domain (of appropriate arity)

# Semantics in First-order Logic

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## Definition

An **atomic sentence**

$$\textit{predicate} ( \textit{term}_1, \dots, \textit{term}_n )$$

is true in a certain model (that consists of a domain and an interpretation)

iff

the domain elements that are the interpretations of  $\textit{term}_1, \dots, \textit{term}_n$   
are in the relation that is the interpretation of  $\textit{predicate}$



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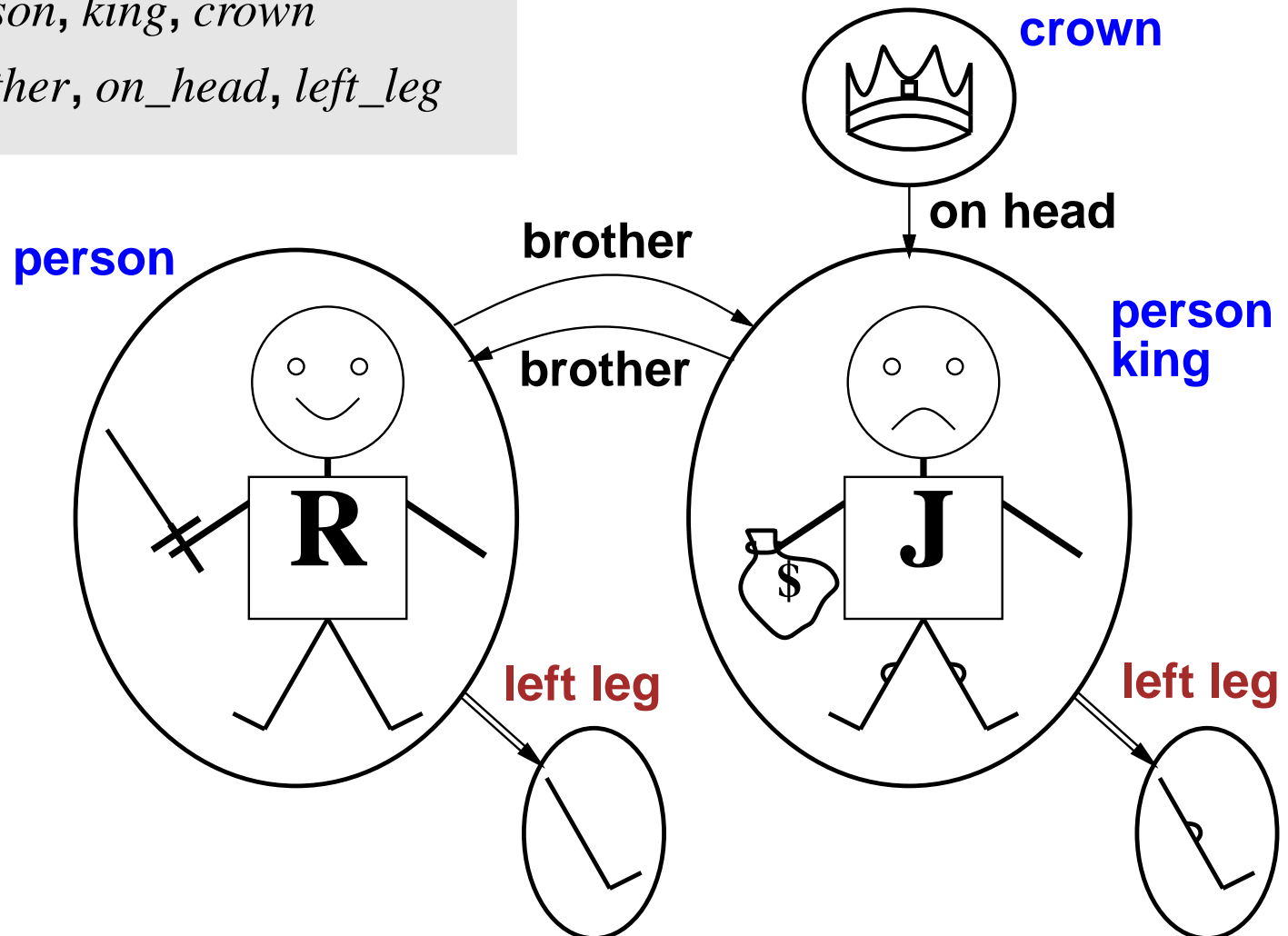
iff

the domain elements that are the interpretations of  $\textit{term}_1, \dots, \textit{term}_n$   
are in the relation that is the interpretation of *predicate*

The truth value of a **complex sentence** in a model  
is computed from the truth-values of its atomic sub-sentences  
in the same way as in propositional logic

# Models for First-order Logic: Example

**Constants:** *KingJohn, Richard*  
**Predicates:** *person, king, crown*  
**Functions:** *brother, on\_head, left\_leg*



# Universal Quantification: Syntax

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## Syntax

$\forall$  *variables sentence*

# Universal Quantification: Syntax

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## Syntax

$\forall$  *variables sentence*

## Example

“Everyone studying in Koblenz is smart:

$$\forall \underbrace{x}_{\text{variables}} \underbrace{(StudiesAt(x, Koblenz) \Rightarrow Smart(x))}_{\text{sentence}}$$

# Universal Quantification: Semantics

---

## Semantics

$\forall xP$  is true in a model

iff

for all domain elements  $d$  in the model:

$P$  is true in the model when  $x$  is interpreted by  $d$

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$\forall xP$  is roughly equivalent to the conjunction of all instances of  $P$

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**Example**  $\forall x \text{StudiesAt}(x, \text{Koblenz}) \Rightarrow \text{Smart}(x)$  equivalent to:

$\text{StudiesAt}(\text{KingJohn}, \text{Koblenz}) \Rightarrow \text{Smart}(\text{KingJohn})$

$\wedge \text{StudiesAt}(\text{Richard}, \text{Koblenz}) \Rightarrow \text{Smart}(\text{Richard})$

$\wedge \text{StudiesAt}(\text{Koblenz}, \text{Koblenz}) \Rightarrow \text{Smart}(\text{Koblenz})$

$\wedge \dots$

# A Common Mistake to Avoid

---

## Note

$\Rightarrow$  is the main connective with  $\forall$

## Common mistake

Using  $\wedge$  as the main connective with  $\forall$



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## Example

**Correct:**  $\forall x (StudiesAt(x, Koblenz) \Rightarrow Smart(x))$

**“Everyone who studies at Koblenz is smart”**

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## Example

**Correct:**  $\forall x (StudiesAt(x, Koblenz) \Rightarrow Smart(x))$

**“Everyone who studies at Koblenz is smart”**

**Wrong:**  $\forall x (StudiesAt(x, Koblenz) \wedge Smart(x))$

**“Everyone studies at Koblenz and is smart”, i.e.,**

**“Everyone studies at Koblenz and everyone is smart”**

# Existential Quantification: Syntax

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## Syntax

$\exists$  *variables sentence*

# Existential Quantification: Syntax

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## Syntax

$\exists$  *variables sentence*

## Example

“Someone studying in Landau is smart:

$$\exists \underbrace{x}_{\text{variables}} \underbrace{(StudiesAt(x, Landau) \wedge Smart(x))}_{\text{sentence}}$$

# Existential Quantification: Semantics

---

## Semantics

$\exists xP$  is true in a model

iff

there is a domain element  $d$  in the model such that:  
 $P$  is true in the model when  $x$  is interpreted by  $d$

# Existential Quantification: Semantics

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## Semantics

$\exists xP$  is true in a model

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there is a domain element  $d$  in the model such that:

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$\exists xP$  is roughly equivalent to the disjunction of all instances of  $P$

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## Intuition

$\exists xP$  is roughly equivalent to the disjunction of all instances of  $P$

**Example**  $\exists x \text{StudiesAt}(x, \text{Landau}) \wedge \text{Smart}(x)$  **equivalent to:**

$\text{StudiesAt}(\text{KingJohn}, \text{Landau}) \wedge \text{Smart}(\text{KingJohn})$

∨  $\text{StudiesAt}(\text{Richard}, \text{Landau}) \wedge \text{Smart}(\text{Richard})$

∨  $\text{StudiesAt}(\text{Landau}, \text{Landau}) \wedge \text{Smart}(\text{Landau})$

∨ ...

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## Example

**Correct:**  $\exists x (StudiesAt(x, Landau) \wedge Smart(x))$

**“There is someone who studies at Landau and is smart”**

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## Note

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## Example

**Correct:**  $\exists x (StudiesAt(x, Landau) \wedge Smart(x))$

**“There is someone who studies at Landau and is smart”**

**Wrong:**  $\exists x (StudiesAt(x, Landau) \Rightarrow Smart(x))$

**“There is someone who, if he/she studies at Landau, is smart”**

**This is true if there is anyone not studying at Landau**

# Properties of Quantifiers

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## Quantifiers of same type commute

$\forall x \forall y$     **is the same as**     $\forall y \forall x$

$\exists x \exists y$     **is the same as**     $\exists y \exists x$

# Properties of Quantifiers

---

Quantifiers of different type do NOT commute

$\exists x \forall y$  is **not** the same as  $\forall y \exists x$

## Example

$\exists x \forall y \text{Loves}(x, y)$

**“There is a person who loves everyone in the world”**

$\forall y \exists x \text{Loves}(x, y)$

**“Everyone in the world is loved by at least one person”**

**(Both hopefully true but different)**

# Properties of Quantifiers

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Quantifiers of different type do NOT commute

$\exists x \forall y$  is **not** the same as  $\forall y \exists x$

## Example

$\forall x \exists y \text{Mother}(x, y)$

“Everyone has a mother” **(correct)**

$\exists y \forall x \text{Mother}(x, y)$

“There is a person who is the mother of everyone” **(wrong)**

# Properties of Quantifiers

---

## Quantifier duality

$\forall x Likes(x, IceCream)$       **is the same as**       $\neg \exists x \neg Likes(x, IceCream)$

$\exists x Likes(x, Broccoli)$       **is the same as**       $\neg \forall x \neg Likes(x, Broccoli)$

# Fun with Sentences

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## ● “Brothers are siblings”

$$\forall x, y (Brother(x, y) \Rightarrow Sibling(x, y))$$

# Fun with Sentences

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- **“Brothers are siblings”**

$$\forall x, y (Brother(x, y) \Rightarrow Sibling(x, y))$$

- **“Sibling” is symmetric**

$$\forall x, y (Sibling(x, y) \Leftrightarrow Sibling(y, x))$$



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- **“Brothers are siblings”**

$$\forall x, y (Brother(x, y) \Rightarrow Sibling(x, y))$$

- **“Sibling” is symmetric**

$$\forall x, y (Sibling(x, y) \Leftrightarrow Sibling(y, x))$$

- **“One’s mother is one’s female parent”**

$$\forall x, y (Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y)))$$

# Fun with Sentences

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- **“Brothers are siblings”**

$$\forall x, y (Brother(x, y) \Rightarrow Sibling(x, y))$$

- **“Sibling” is symmetric**

$$\forall x, y (Sibling(x, y) \Leftrightarrow Sibling(y, x))$$

- **“One’s mother is one’s female parent”**

$$\forall x, y (Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y)))$$

- **“A first cousin is a child of a parent’s sibling”**

$$\forall x, y (FirstCousin(x, y) \Leftrightarrow \exists p, ps (Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)))$$

# Equality

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## Semantics

$term_1 = term_2$  is true under a given interpretation

**if and only if**

$term_1$  and  $term_2$  have the same interpretation

# Equality

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## Example

**Definition of (full) sibling in terms of *Parent***

$$\begin{aligned} \forall x, y \text{ Sibling}(x, y) \Leftrightarrow & (\neg(x = y) \wedge \\ & \exists m, f (\neg(m = f) \wedge \\ & \text{Parent}(m, x) \wedge \text{Parent}(f, x) \wedge \\ & \text{Parent}(m, y) \wedge \text{Parent}(f, y))) \end{aligned}$$

# Properties of First-order Logic

---

## Important notions

- **validity**
- **satisfiability**
- **unsatisfiability**
- **entailment**

**are defined for first-order logic in the same way as for propositional logic**

# Properties of First-order Logic

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## Important notions

- validity
- satisfiability
- unsatisfiability
- entailment

are defined for first-order logic in the same way as for propositional logic

## Calculi

There are sound and complete calculi for first-order logic (e.g. resolution)

- Whenever  $KB \vdash \alpha$ , it is also true that  $KB \models \alpha$
- Whenever  $KB \models \alpha$ , it is also true that  $KB \vdash \alpha$

But these calculi **CANNOT decide** validity, entailment, etc.

# Properties of First-order Logic

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## In propositional logic

Validity, satisfiability, unsatisfiability are **decidable**

## In first-order logic

The set of valid, and the set of unsatisfiable formulas are **enumerable**

The set of satisfiable formulas is **NOT EVEN enumerable**