Propositional Logic (a.k.a. Sentential Logic)

Boolean Operators

- Propositional logic (sentence logic) dealt quite satisfactorily with sentences using conjunctive words (접속사) like not, and, or, and if … then.
- A proposition (*p*, *q*, *r*, …) in a propositional calculus can get a boolean value (i.e. true or false)
- Propositional formula can be built by combining smaller formula with boolean operators such as ¬, ∧, ∨
- How many different unary boolean operators exist?

p	o_1	o_2	o_3	o_4	…
T	T	T	F	F	
F	T	F	T	F	

- How many different binary boolean operators exist?

Binary Boolean Operators

x_1	x_2	$\circ_1$	$\circ_2$	$\circ_3$	$\circ_4$	$\circ_5$	$\circ_6$	$\circ_7$	$\circ_8$
T	T	T	T	T	T	T	T	T	T
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

x_1	x_2	$\circ_9$	$\circ_{10}$	$\circ_{11}$	$\circ_{12}$	$\circ_{13}$	$\circ_{14}$	$\circ_{15}$	$\circ_{16}$
T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

Boolean Operators

op	name	symbol	op	name	symbol
o_2	disjunction	$\lor$	o_{15}	nor	$\downarrow$
o_8	conjunction	$\land$	o_9	nand	$\uparrow$
o_5	implication	$\rightarrow$	o_{12}		
o_3	reverse implication	$\leftarrow$	o_{14}		
o_7	equivalence	$\leftrightarrow$	o_{10}	exclusive or	$\oplus$

Ambiguous representation of formulas

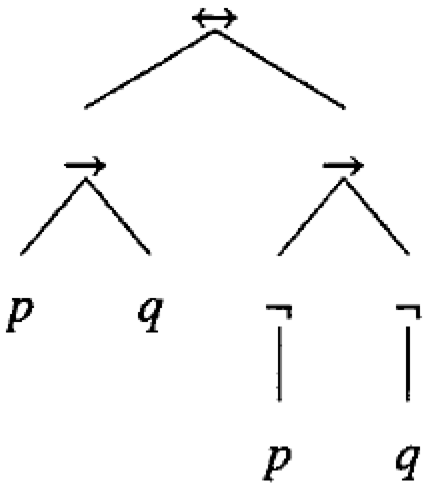


Figure 2.3 Formation tree for $p \rightarrow q \leftrightarrow \neg p \rightarrow \neg q$

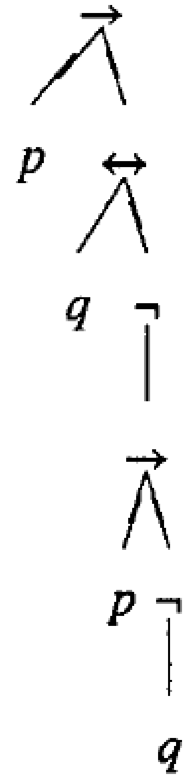


Figure 2.4 Another formation tree

Other ways to remove ambiguity

- Use parenthesis
- Define precedence and associativity
  - The precedence order
    - $\neg > \wedge > \uparrow > \vee > \downarrow > \rightarrow > \leftrightarrow$
  - Operators are assumed to associate to the right
    - $a \rightarrow b \rightarrow c$ means $(a \rightarrow (b \rightarrow c))$
    - $a \vee b \vee c$ means $(a \vee (b \vee c))$
      - Some textbook considers $\wedge$, $\vee$, $\leftrightarrow$ as associate to the left. So be careful.

Assignment/ Interpretation ν

$f = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_4)$

Assignment ν_1 to ν_5, ν_7, ν_9, ν_9, ν_{10}, and ν_{13} to ν_{16} which evaluates *f* as true are called as model or solution of *f*

Model/ solution

ν	x_1	x_2	x_3	x_4	*f*
ν_1	T	T	T	T	T
ν_2	T	T	T	F	T
ν_3	T	T	F	T	T
ν_4	T	T	F	F	T
ν_5	T	F	T	T	T
ν_6	T	F	T	F	F
ν_7	T	F	F	T	T
ν_8	T	F	F	F	F
ν_9	F	T	T	T	T
ν_{10}	F	T	T	F	T
ν_{11}	F	T	F	T	F
ν_{12}	F	T	F	F	F
ν_{13}	F	F	T	T	T
ν_{14}	F	F	T	F	T
ν_{15}	F	F	F	T	T
ν_{16}	F	F	F	F	T

Interpretations

- Inductive truth value calculation for given formula A

A	$v(A_1)$	$v(A_2)$	$v(A)$
$\neg A_1$	T		F
$\neg A_1$	F		T
$A_1 \vee A_2$	F	F	F
$A_1 \vee A_2$	otherwise		T
$A_1 \wedge A_2$	T	T	T
$A_1 \wedge A_2$	otherwise		F
$A_1 \rightarrow A_2$	T	F	F
$A_1 \rightarrow A_2$	otherwise		T

A	$v(A_1)$	$v(A_2)$	$v(A)$
$A_1 \uparrow A_2$	T	T	F
$A_1 \uparrow A_2$	otherwise		T
$A_1 \downarrow A_2$	F	F	T
$A_1 \downarrow A_2$	otherwise		F
$A_1 \leftrightarrow A_2$	$v(A_1) = v(A_2)$		T
$A_1 \leftrightarrow A_2$	$v(A_1) \neq v(A_2)$		F
$A_1 \oplus A_2$	$v(A_1) \neq v(A_2)$		T
$A_1 \oplus A_2$	$v(A_1) = v(A_2)$		F

Figure 2.5 Evaluation of truth values of formulas

Examples

Example 2.7 Let $A = (p \to q) \leftrightarrow (\neg q \to \neg p)$, and let v the assignment such that $v(p) = F$ and $v(q) = T$, and $v(p_i) = T$ for all other $p_i \in \mathcal{P}$. Extend v to an interpretation. The truth value of A can be calculated inductively using Figure 2.5:

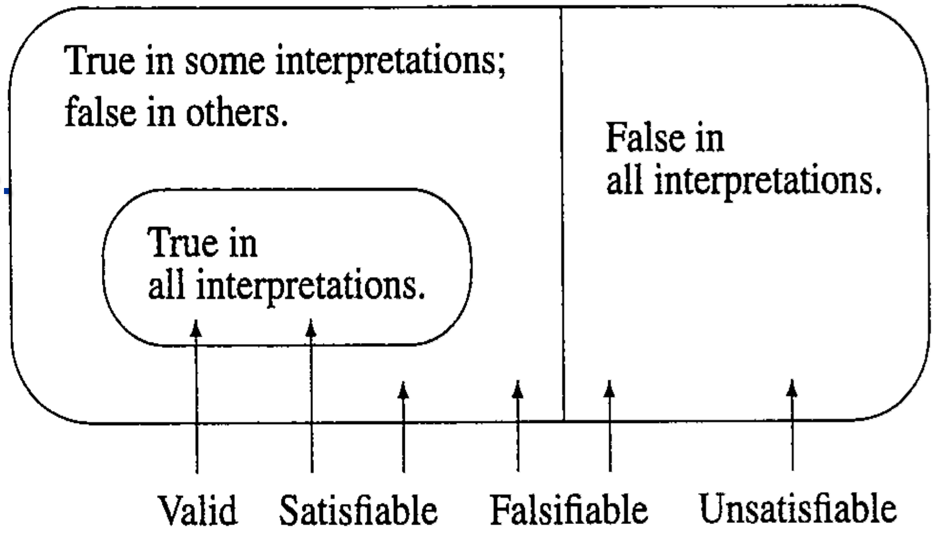
$$
\begin{aligned}
v(p \to q) &= T \\
v(\neg q) &= F \\
v(\neg p) &= T \\
v(\neg q \to \neg p) &= T \\
v((p \to q) \leftrightarrow (\neg q \to \neg p)) &= T.
\end{aligned}
$$

Example 2.8 $v(p \to (q \to p)) = T$ but $v((p \to q) \to p) = F$ under the above interpretation, emphasizing that the linear string $p \to q \to p$ is ambiguous. ▯

Example 2.12 Let $S = \{p \to q,\ p,\ p \vee s \leftrightarrow s \wedge q\}$, and let v be the assignment given by $v(p) = T$, $v(q) = F$, $v(r) = T$, $v(s) = T$. v is an interpretation for S and assigns the truth values

Satisfiability v.s. validity

- Definition 2.24
  - A propositional formula *A* is satisfiable iff $\nu(A)=T$ for some interpretation ν.
    - A satisfying interpretation is called a model for *A*.
  - *A* is valid, denoted $\models A$, iff $\nu(A) = T$ for all interpretation ν.
    - A valid propositional formula is also called a tautology.
- Theorem 2.25
  - *A* is valid iff $\neg A$ is unsatisfiable.
  - *A* is satisfiable iff $\neg A$ is falsifiable.



Satisfiability v.s. validity

- Example 2.27 Is $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$ valid?

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*F*	*T*
F	*T*	*T*	*T*	*T*
F	*F*	*T*	*T*	*T*

- Example 2.28 $p \vee q$ is satisfiable but not valid