HW #3: Due Nov 14th 11:00 AM

1. Let m be a constant, f a function symbol with one argument and S and B two predicate symbols, each with two arguments. Which of the following strings are formulas in predicate logic? Specify a reason for failure for strings which aren’

   (a) $S(m, x)$
   (b) $B(m, f(m))$
   (c) $f(m)$
   (d) $B(B(m, x), y)$
   (e) $S(B(m), z)$
   (f) $(B(x, y) \to (\exists z\, S(z, y)))$
   (g) $(S(x, y) \to S(y, f(f(x)))$
   (h) $(B(x) \to B(B(x)))$.

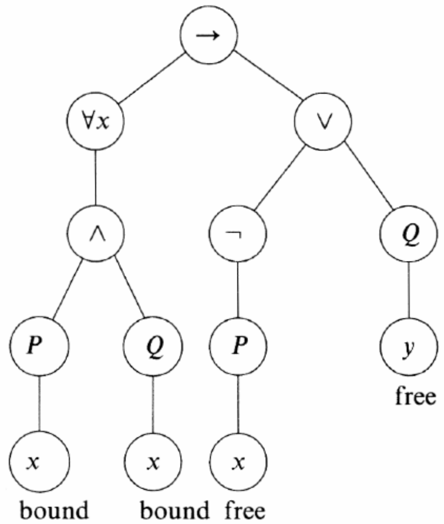

Fig. 2.3. A parse tree of a predicate logic formula illustrating *free* and *bound* occurrences of variables.

2. Use the predicates

$$\begin{aligned} A(x,y):&\quad x \text{ admires } y\\ B(x,y):&\quad x \text{ attended } y\\ P(x):&\quad x \text{ is a professor}\\ S(x):&\quad x \text{ is a student}\\ L(x):&\quad x \text{ is a lecture} \end{aligned}$$

and the function symbol (= constant)

$$m: \quad \text{Mary}$$

to translate the following into predicate logic:

(a) Mary admires every professor.
(The answer is *not* $\forall x\, A(m, P(x))$; see exercise 1.)
(b) Some professor admires Mary.
(c) Mary admires herself.
(d) No student attended every lecture.
(e) No lecture was attended by every student.
(f) No lecture was attended by any student.

3. Let c and d be constants, f a function symbol with one argument, g a function symbol with two arguments and h a function symbol with three arguments. Further, P and Q are predicate symbols with three arguments. Which of the following strings are formulas in predicate logic? Specify a reason for failure for strings which aren't. Draw parse trees of all strings which are formulas of predicate logic.

(a) $\forall x\, P(f(d), h(g(c,x), d, y))$
(b) $\forall x\, P(f(d), h(P(x,y), d, y))$
(c) $\forall x\, Q(g(h(x, f(d), x), g(x,x)), h(x,x,x), c)$
(d) $\exists z\, (Q(z,z,z) \to P(z))$
(e) $\forall x\, \forall y\, (g(x,y) \to P(x,y,x))$
(f) $Q(c,d,c)$.

4. Use the predicate specifications

$$\begin{aligned} B(x,y):&\quad x \text{ beats } y\\ F(x):&\quad x \text{ is an (American) football team}\\ Q(x,y):&\quad x \text{ is quarterback of } y\\ L(x,y):&\quad x \text{ loses to } y \end{aligned}$$

and the constant symbols

$$\begin{aligned} c:&\quad \text{Wildcats}\\ j:&\quad \text{Jayhawks} \end{aligned}$$

to translate the following into predicate logic.

(a) Every football team has a quarterback.
(b) If the Jayhawks beat the Wildcats, then the Jayhawks do not lose to every football team.
(c) The Wildcats beat some team, which beat the Jayhawks.

5. Find appropriate predicates and their specification to translate the following into predicate logic:

(a) All red things are in the box.
(b) Only red things are in the box.
(c) No animal is both a cat and a dog.
(d) Every prize was won by a boy.
(e) A boy won every prize.

6. Let $F(x,y)$ mean that x is the father of y; $M(x,y)$ denotes x is the mother of y. Similarly, $H(x,y)$, $S(x,y)$, and $B(x,y)$ say that x is the husband/sister/brother of y, respectively. You may also use constants to denote individuals, like 'Ed' and 'Patsy'. However, you are not allowed to use any predicate symbols other than the above to translate the following sentences into predicate logic:

(a) Everybody has a mother.
(b) Everybody has a father and a mother.
(c) Whoever has a mother has a father.
(d) Ed is a grandfather.
(e) All fathers are parents.
(f) All husbands are spouses.
(g) No uncle is an aunt.
(h) All brothers are siblings.
(i) Nobody's grandmother is anybody's father.
(j) Ed and Patsy are husband and wife.
(k) Carl is Monique's brother-in-law.

7. Formalise the following sentences in predicate logic, defining predicate symbols as appropriate:

(a) Everybody who visits New Orleans falls in love with it.
(b) There is a trumpet player who lives in New Orleans, but who does not like crawfish étouffée.
(c) There are at least two saxophone players who were born in New Orleans and who play better than every sax player in New York city.

8. Consider the formula

$$\phi \stackrel{\text{def}}{=} \forall x \forall y\, Q(g(x,y), g(y,y), z).$$

(Note that an environment/lookup table l means anassignment $\sigma_{\mathcal{I}}$)

Obviously, Q is a predicate with three arguments and g a function with two arguments. Find two models $\mathcal{M}$ and $\mathcal{M}'$ with respective environments l and l' such that $\mathcal{M} \vDash_l \phi$ but $\mathcal{M}' \nvDash_{l'} \phi$.

9. Consider the sentence

$$\phi \stackrel{\text{def}}{=} \forall x \exists y \exists z\, (P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x))).$$

Which of the following models satisfies ϕ?

(consider a model $\mathcal{M}$ is similar to an interpretation $\mathcal{I}$)

(a) The model $\mathcal{M}$ consists of the set of natural numbers with $P^{\mathcal{M}} \stackrel{\text{def}}{=} \{(m,n) \mid m < n\}$.

(b) The model $\mathcal{M}'$ consists of the set of natural numbers with $P^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(m, 2 * m) \mid m \text{ natural number}\}$.

(c) The model $\mathcal{M}''$ consists of the set of natural numbers with $P^{\mathcal{M}''} \stackrel{\text{def}}{=} \{(m,n) \mid m < n+1\}$.

10. Let P be a predicate with two arguments. Find a model $\mathcal{M}$ which satisfies the sentence $\forall x \neg P(x,x)$. Find also a model $\mathcal{M}'$ such that $\mathcal{M}' \nvDash \forall x \neg P(x,x)$.

11. Consider the sentence $\forall x(\exists y P(x,y) \wedge (\exists z P(z,x) \rightarrow \forall y P(x,y)))$. We already noted that its meaning in a given model is independent of the chosen look-up table l. Please simulate the evaluation of this sentence in a model of your choice, focusing on how the initial look-up table l grows and shrinks like a stack when you evaluate its subformulas according to the definition of the satisfaction relation.

12. Let $\mathcal{F} \stackrel{\text{def}}{=} \{d, f, g\}$, where d is a constant symbol, f a function symbol with three arguments and g a function symbol with two arguments. As model $\mathcal{M}$, we choose the set of natural numbers $0, 1, 2, \ldots$. Further, $d^{\mathcal{M}} \stackrel{\text{def}}{=} 2$, $f^{\mathcal{M}}(k,n,m) \stackrel{\text{def}}{=} k * n + m$ and $g^{\mathcal{M}}(k,n) \stackrel{\text{def}}{=} k + n * n$. E.g. $f^{\mathcal{M}}(1,2,3)$ equals 5 and $g^{\mathcal{M}}(2,3)$ equals 11. Assuming a look-up table l with $l(x) \stackrel{\text{def}}{=} 5$ and $l(y) \stackrel{\text{def}}{=} 7$, compute the meaning of the terms below in the model $\mathcal{M}$:

(note that a function with no parameter can be considered as a constant)

* (a) $f(d, x, d)$

(b) $f(g(x,d), y, g(d,d))$

(c) $g(f(g(d,y), f(x, g(d,d), x), y), f(y, g(d,d), d))$.

13. Let ϕ be the formula

$$\forall x \forall y \exists z\, (R(x,y) \rightarrow R(y,z)),$$

(note that A is a target domain of $\mathcal{M}$)

where R is a predicate symbol of two arguments.

* (a) Let $A \stackrel{\text{def}}{=} \{a,b,c,d\}$ and $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(b,c),(b,b),(b,a)\}$. Do we have $\mathcal{M} \vDash \phi$? Justify your answer, whatever it is.

* (b) Let $A' \stackrel{\text{def}}{=} \{a,b,c\}$ and $R^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(b,c),(a,b),(c,b)\}$. Do we have $\mathcal{M}' \vDash \phi$? Justify your answer, whatever it is.

14. We call a set $\phi_1, \phi_2, \ldots, \phi_n$ of formulas *consistent* if there is a model of all predicate and function symbols involved such that all formulas $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to T for that model. For each set of formulas below show that they are consistent:

(a) $\forall x \neg S(x,x)$, $\exists x P(x)$, $\forall x \exists y S(x,y)$, $\forall x (P(x) \rightarrow \exists y S(y,x))$

* (b) $\forall x \neg S(x,x)$, $\forall x \exists y S(x,y)$, $\forall x \forall y \forall z ((S(x,y) \wedge S(y,z)) \rightarrow S(x,z))$

(c) $(\forall x (P(x) \vee Q(x))) \rightarrow \exists y R(y)$, $\forall x (R(x) \rightarrow Q(x))$, $\exists y (\neg Q(y) \wedge P(y))$

* (d) $\exists x S(x,x)$, $\forall x \forall y (S(x,y) \rightarrow (x = y))$.

15. Let P and Q be predicate symbols with one argument each. For the formulas below, check whether they are **valid**. If not, then you have to find a model of P and Q such that the formula evaluates to F.

(a) $\forall x \forall y ((P(x) \rightarrow P(y)) \wedge (P(y) \rightarrow P(x)))$

(b) $(\forall x ((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x)))) \rightarrow ((\forall x P(x)) \rightarrow (\forall x Q(x)))$

(c) $((\forall x P(x)) \rightarrow (\forall x Q(x))) \rightarrow (\forall x ((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x))))$

(d) (Difficult.) $(\forall x \exists y (P(x) \rightarrow Q(y))) \rightarrow (\exists y \forall x (P(x) \rightarrow Q(y)))$.

16. For each of the formulas of predicate logic below, either find a model which does not satisfy it, or prove it

(a) $(\forall x \forall y (S(x,y) \rightarrow S(y,x))) \rightarrow (\forall x \neg S(x,x))$

(b) $\exists y ((\forall x P(x)) \rightarrow P(y))$

(c) $(\forall x (P(x) \rightarrow \exists y Q(y))) \rightarrow (\forall x \exists y (P(x) \rightarrow Q(y)))$

(d) $(\forall x \exists y (P(x) \rightarrow Q(y))) \rightarrow (\forall x (P(x) \rightarrow \exists y Q(y)))$

(e) $\forall x \forall y (S(x,y) \rightarrow (\exists z (S(x,z) \wedge S(z,y))))$

(f) $(\forall x \forall y (S(x,y) \rightarrow (x = y))) \rightarrow (\forall z \neg S(z,z))$

- 17. Let $\phi = \forall xy \exists z\, [(x \leq z \wedge y \leq z) \vee (z \leq x \wedge z \leq y)]$. Find posets $\mathcal{U}_1$ and $\mathcal{U}_2$ s.t. $\mathcal{U}_1 \models \phi$ and $\mathcal{U}_2 \models \neg\phi$
- 18. Make predicate formulas which define a group. Group is defined as follows
  - The bracketing is unimportant (associativity): $a \times (b \times c) = (a \times b) \times c$
  - There is an element that does not cause anything to happen (identity element): $a \times 1 = 1 \times a = a$
  - Each element a has a "mirror image" (inverse element) a^{-1} that has the property to yield the identity element when combined with a:

    $a \times a^{-1} = a^{-1} \times a = 1$
- 19. Let $\mathcal{I}_1 = (\mathcal{N}, \{<\}, \{0\})$ and $\mathcal{I}_2 = (\mathcal{N}, \{\triangle\}, \{0\}\}$ where $n \triangle m$ iff
  - $n < m$ and n, m both even or both odd, or
  - if n is even and m odd

  Give a predicate formula ϕ s.t. $\mathcal{I}_1 \models \phi$ and $\mathcal{I}_2 \models \neg\phi$