Predicate Calculus - Semantics 1/4

Moonzoo Kim
CS Division of EECS Dept.
KAIST

moonzoo@cs.kaist.ac.kr
http://pswlab.kaist.ac.kr/courses/cs402-07

Introduction to predicate calculus (1/2)

- Propositional logic (sentence logic) dealt quite satisfactorily with sentences using conjunctive words (접속사) like not, and, or, and if … then. But it fails to reflect the *finer logical structure* of the sentence
- What can we reason about a sentence itself which deals with its target domain?
  - ex. Jane is taller than Alice (target domain : human being)
  - ex2. For natural numbers x and y, $x+y \geq -(x+y)$ (target domain: $\mathcal{N}$)
- What can we reason about a sentence itself which also deal with modifiers like there exists…, all …, among … and only …. ?
  - Note that these modifiers enable us to reason about an infinite domain because we do not have to enumerate all elements in the domain

Introduction to predicate calculus (2/2)

- Ex. Every student is younger than some instructor
  - We could simply identify this assertion with a propositional atom p. However, this fails to reflect the finer logical structure of this sentence
- This statement is about being a student, being an instructor and being younger than somebody else for a set of university members as a target domain
  - We need to express them and use predicates for this purpose
  - S(yunho), I(moonzoo), Y(yunho,moonzoo)
- We need variables x, y to not to write down all instance of S(-), I(-), Y(-)
  - Every student x is young than some instructor y
- Finally, we need quantifiers to capture the actual elements by variables
  - For every x, if x is a student, then there is some y which is an instructor such that x is younger than y
  - $\forall x\ (S(x) \to (\exists y\ (I(y) \land Y(x,y))))$

Relations and predicates

- The axioms and theorems of mathematics are defined on arbitrary sets (domain) such as the set of integers $\mathcal{Z}$
  - ex. Fermat's last theorem
    - If an integer *n* is greater than 2, then the equation $a^n + b^n = c^n$ has no solutions in non-zero integers *a*, *b*, and *c*.
  - Can we express the Fermat's last theorem in propositional logic?
- The predicate calculus extends the propositional calculus with predicate letters that are interpreted as relations on a domain
  - i.e., predicates are interpreted upon domain
- Def 5.2. A relation can be represented by a boolean valued function $R:D^n \rightarrow \{T,F\}$, by mapping an n-tuple to T iff it is included in the relation
  - $R(d_1,\ldots d_n) = T$ iff $(d_1,\ldots d_n) \in \mathcal{R}$

Predicate formulas

- Let $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{V}$ be countable sets of symbols called predicate letters, constants, and variables, respectively.
  - $\mathcal{P}$={p,q,r} $\mathcal{A}$={a,b,c}, $\mathcal{V}$={x,y,z}
- Def 5.4 Atomic formulas and formulas
  - atomic formula
    - argument ::= x for any x $\in \mathcal{V}$
    - argument ::= a for any a $\in \mathcal{A}$
    - argument_list ::= argument$^+$
    - atomic_formula ::= p |
      p(argument_list) for any p $\in \mathcal{P}$
  - formula ::== atomic_formula
  - formula ::= $\neg$ formula
  - formula ::= formula $\vee$ formula
  - formula ::= $\forall$ x formula
  - formula ::= $\exists$ x formula

1. $\forall x \forall y (p(x, y) \rightarrow p(y, x))$.
2. $\forall x \exists y p(x, y)$.
3. $\exists x \exists y (p(x) \wedge \neg p(y))$.
4. $\forall x p(a, x)$.
5. $\forall x (p(x) \wedge q(x)) \leftrightarrow (\forall x p(x) \wedge \forall x q(x))$.
6. $\exists x (p(x) \vee q(x)) \leftrightarrow (\exists x p(x) \vee \exists x q(x))$.
7. $\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$.
8. $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x))$.

Free and bound variables

- Def 5.6
  - $\forall$ is the universal quantifier and is read 'for all'.
  - $\exists$ is the existential quantifier and is read 'there exists'.
  - In a quantified formula $\forall xA$, x is the quantified variable and A is the scope of the quantified variable.
- Def 5.7 Let A be a formula. An occurrence of a variable x in A is a free variable of A iff x is not within the scope of a quantified variable x.
  - Notation: $A(x_1,\ldots x_n)$ indicates that the set of free variables of the formula A is a subset of $\{x_1,\ldots x_n\}$. A variable which is not free is bound.
  - If a formula has no free variable it is closed
  - If $\{x_1,\ldots,x_n\}$ are all the free variables of A, the universal closure of A is $\forall x_1 \ldots \forall x_n A$ and the existential closure is $\exists x_1 \ldots \exists x_n A$
- Ex 5.8 $p(x,y)$, $\exists y\, p(x,y)$, $\forall x \exists y\, p(x,y)$
- Ex 5.9
  - In $(\forall x\, p(x)) \wedge q(x)$, the occurrence of x in p(x) is bound and the occurrence in q(x) is free. The universal closure is $\forall x\, (\forall x p(x) \wedge q(x))$.
  - Obviously, it would have been better to write the formula as $\forall x p(x) \wedge q(y)$ where y is the free variable

Interpretations (1/5)

- Def 5.10 Let U be a set of formulas s.t. $\{p_1,\ldots p_m\}$ are all the predicate letters and $\{a_1,\ldots, a_k\}$ are all the constant symbols appearing in U. An interpretation $\mathcal{I}$ is a triple (D, $\{R_1,\ldots R_m\}$, $\{d_1,\ldots,d_k\}$), where
  - D is a non-empty set,
  - R_i is an n_i-ary relation on D that is assigned to the n_i-ary predicate p_i
    - Notation: $p_i^{\mathcal{I}} = R_i$
  - $d_i \in D$ is an element of D that is assigned to the constant a_i
    - Notation: $a_i^{\mathcal{I}} = d_i$
- Ex 5.11. Three numerical interpretations for $\forall x\ p(a,x)$:
  - $\mathcal{I}_1 = (\mathcal{N}, \{\leq\}, \{0\})$, $\mathcal{I}_2 = (\mathcal{N}, \{\leq\}, \{1\})$. $\mathcal{I}_3 = (\mathcal{Z}, \{\leq\}, \{0\})$.
  - $\mathcal{I}_4 = (\mathcal{S}$, {substr},{“”}) where S is the set of strings on some alphabet

Interpretations (2/5)

- Def 5.12 Let $\mathcal{I}$ be an interpretation. An assignment $\sigma_{\mathcal{I}} : \mathcal{V} \rightarrow D$ is a function which maps every variable to an element of the domain of $\mathcal{I}$. $\sigma_{\mathcal{I}}[x_i \leftarrow d_i]$ is an assignment that is the same as $\sigma_{\mathcal{I}}$ except that x_i is mapped to d_i
- Def 5.13 Let A be a formula, $\mathcal{I}$ an interpretation and $\sigma_{\mathcal{I}}$ an assignment. $v_{\sigma_{\mathcal{I}}}(A)$, the truth value of A under $\sigma_{\mathcal{I}}$ is defined by induction on the structure of A:
  - Let $A = p_k(c_1,\ldots,c_n)$ be an atomic formula where each c_i is either a variable x_i or a constant a_i. $v_{\sigma_{\mathcal{I}}}(A) = T$ iff
    - $\langle d_1,\ldots d_n \rangle \in R_k$ where R_k is the relation assigned by $\mathcal{I}$ to p_k and
    - d_i is the domain element assigned to c_i, either
      - by $\mathcal{I}$ if c_i is a constant or
      - by $\sigma_{\mathcal{I}}$ if c_i is variable
  - $v_{\sigma_{\mathcal{I}}}(\neg A) = T$ iff $v_{\sigma_{\mathcal{I}}}(A) = F$
  - $v_{\sigma_{\mathcal{I}}}(A_1 \vee A_2)$ iff $v_{\sigma_{\mathcal{I}}}(A_1) = T$ or $v_{\sigma_{\mathcal{I}}}(A_2) = T$
  - $v_{\sigma_{\mathcal{I}}}(\forall x\, A_1) = T$ iff $v_{\sigma_{\mathcal{I}}[x \leftarrow d]}(A_1) = T$ for all $d \in D$
  - $v_{\sigma_{\mathcal{I}}}(\exists x\, A_1) = T$ iff $v_{\sigma_{\mathcal{I}}[x \leftarrow d]}(A_1) = T$ for some $d \in D$

Interpretations (3/5)

- Thm 5.14 Let A be a closed formula. Then $v_{\sigma_{\mathcal{I}}}(A)$ does not depend on $\sigma_{\mathcal{I}}$. In such cases, we use simply $v_{\mathcal{I}}(A)$ instead of $v_{\sigma_{\mathcal{I}}}(A)$
- (important!) Thm 5.15 Let A' = $A(x_1,\ldots,x_n)$ be a non-closed formula and let $\mathcal{I}$ be an interpretation. Then:
  - $v_{\sigma_{\mathcal{I}}}(A')=T$ for some assignment $\sigma_{\mathcal{I}}$ iff $v_{\mathcal{I}}(\exists x_1 \ldots \exists x_n\ A') = T$
  - $v_{\sigma_{\mathcal{I}}}(A')=T$ for all assignment $\sigma_{\mathcal{I}}$ iff $v_{\mathcal{I}}(\forall x_1 \ldots \forall x_n\ A') = T$
  - Thm 5.15 is important since we have many chances to add or remove quantified variables to and from formula during proofs.
- Def 5.16 A closed formula A is true in $\mathcal{I}$ or $\mathcal{I}$ is a model for A, if $v_{\mathcal{I}}(A) = T$.
  - Notation: $\mathcal{I} \vDash A$
  - Note that we overload $\vDash$ with usual logical consequence as in propositional logic
    - $\{A_1, A_2, A_3\} \vDash A$
- Def 5.18 A closed formula A is satisfiable if for some interpretation $\mathcal{I}$, $\mathcal{I} \vDash A$. A is valid if for all interpretations $\mathcal{I}$, $\mathcal{I} \vDash A$
  - Notation: $\vDash A$.

Interpretation (4/5)

- Ex 5.19 $\models (\forall x\ p(x)) \rightarrow p(a)$
  - Suppose that it is not. Then there must be an interpretation $\mathcal{I}$ = (D, {R}, {d}) such that $v_{\mathcal{I}}(\forall x\ p(x))$ = T and $v_{\mathcal{I}}(p(a))$ = F
  - By Thm 5.15, $v_{\sigma_{\mathcal{I}}}(p(x))$ = T for all assignments $\sigma_{\mathcal{I}}$, in particular for the assignment $\sigma'_{\mathcal{I}}$ that assigns d to x. But p(a) is closed, so $v_{\sigma_{\mathcal{I}}}(p(a)) = v_{\mathcal{I}}(p(a))$ = F, a contradiction

Example 5.20 Here is a semantic analysis of the formulas from Example 5.5:

- $\forall x \forall y (p(x, y) \rightarrow p(y, x))$
  The formula is satisfiable in an interpretation where p is assigned a symmetric relation like $=$.
- $\forall x \exists y p(x, y)$
  The formula is satisfiable in an interpretation where p is assigned a relation that is a total function, such as $(x, y) \in R$ iff $y = x + 1$ for $x, y \in \mathcal{Z}$.
- $\exists x \exists y (p(x) \wedge \neg p(y))$
  This formula is satisfiable only in a domain with at least two elements.

Interpretation (5/5)

- $\forall x p(a, x)$

  This expresses the existence of a special element. For example, if p is interpreted by the relation $\leq$ on the domain $\mathcal{N}$, then the formula is true for $a = 0$. If we change the domain to $\mathcal{Z}$ the formula is false for the same assignment of $\leq$ to p. Thus a change of domain alone can falsify a formula.

- $\forall x(p(x) \wedge q(x)) \leftrightarrow (\forall x p(x) \wedge \forall x q(x))$

  The formula is valid. We prove the forward direction and leave the converse as an exercise. Let $\mathcal{I} = (D, \{R_1, R_2\}, \{\,\})$ be an arbitrary interpretation. By Theorem 5.15, $v_{\sigma_\mathcal{I}}(p(x) \wedge q(x)) = T$ for all all assignments $\sigma_\mathcal{I}$, and by the inductive definition of an interpretation, $v_{\sigma_\mathcal{I}}(p(x)) = T$ and $v_{\sigma_\mathcal{I}}(q(x)) = T$ for all assignments $\sigma_\mathcal{I}$. Again by Theorem 5.15, $v_\mathcal{I}(\forall x p(x)) = T$ and $v_\mathcal{I}(\forall x q(x)) = T$, and by the definition of interpretation $v_\mathcal{I}(\forall x p(x) \wedge \forall x q(x)) = T$.

  Show that $\forall$ does not distribute over disjunction by constructing a falsifying interpretation for $\forall x(p(x) \vee q(x)) \leftrightarrow (\forall x p(x) \vee \forall x q(x))$.

- $\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$

  This is a valid formula, but its converse is not.