Predicate Calculus
- Undecidability of predicate calculus

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Undecidable problems

- Can you tell whether or not the following program halts?

```
/* Fermat's last theorem: for n > 2, there
   exists no positive integers x,y,z s.t. x^n +
   y^n = z^n */
main() {
    Nat n, total, x, y, z;
    scanf('%d",&n);
    total=3;
    while(1) {/* loop invariant:total= x+y+z*/
        for(x=1; x<= total-2; x++) {
            for(y=1; y <= total-x-1; y++) {
                z= total - x -y;
                if(x^n + y^n == z^n)) halt;
            }
        }
        total++;}}
```

- It would be remarkable indeed if we could make an algorithm that could examine any program P and tell whether P would halt.
  - In other words, to decide whether a given program halts or not is, at least, as hard as proving the Fermat's last theorem which took 300 years
- We know that no such algorithm exists –
  - Halting problem is undecidable

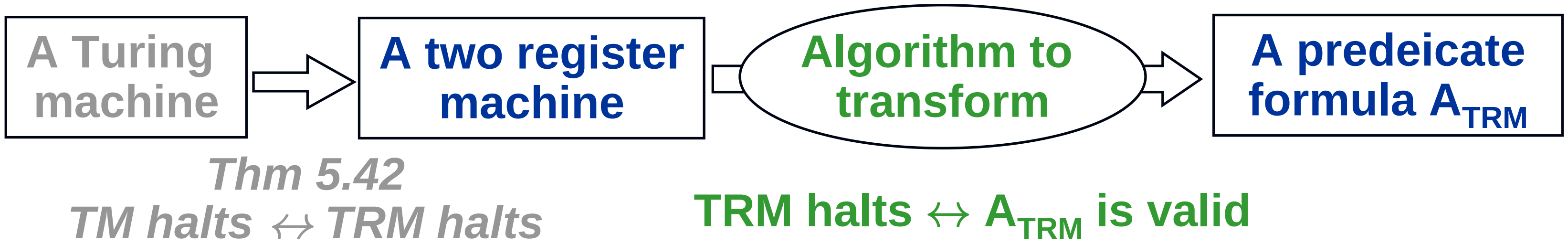
Transform of the Halting problem (1/2)

- It is undecidable to check whether a Turing machine (TM) will halt if started on a blank tape (halting problem)
- To prove the undecidability of predicate logic, we give an algorithm which produces a formula A_{TM} in the predicate calculus for every Turing machine, s.t. A_{TM} is valid iff a Turing machine halts
  - Note that we do not make a Turing machine M for every predicate formula, since it is enough to show that checking some predicate formulas is undecidable
- If we have such an algorithm, it is clear that validity check of predicate formula is at least as hard as halting problem (i.e., undecideble)

TM halts $\leftrightarrow$ A_{TM} is valid

Transform of the Halting problem (2/2)

- To simplify the proof of the transformation algorithm, we work with two-register machines (TRM) rather than directly with Turing machine
  - i.e., we will show there exists such A_{TRM} for a two-register machine
- Thm 5.42 Given a Turing machine that computes a function f, a two-register machine can be constructed to compute the same function f



A two-register machine M

- Def 5.41 A two-register machine M consists of two registers x and y which can hold natural numbers, and a program $P = (L_0,\ldots,L_n)$ which is a list of instructions. L_n is the instruction `halt`, and for $0 \leq I < n$, L_i is one of:
  - $x := x+1$
  - $y := y+1$
  - if $x = 0$ then goto L_j else $x := x - 1$, $0 \leq j \leq n$
  - if $y = 0$ then goto L_j else $y := y - 1$, $0 \leq j \leq n$
- An execution sequence of M is a sequence of states $s_k = (L_{i_k}, x, y)$, where L_{i_k} is the current instruction at s_k, and x,y are the contents of x and y.
- s_{k+1} is obtained from s_k by executing L_{i_k}.
- The initial state $s_0 = (L_{i_0}, m, 0) = (L_0, m, 0)$ for some m.
- If for some k, $s_k = (L_n, x, y)$, the computation of M has halted and M has computed $y = f(m)$

Examples

/* L1 is executed m times and then this program halts*/
L_0:if x=0 then goto L_2 else x:=x-1
L_1:if y=0 then goto L_0 else y:=y-1
L_2:halt

Execution where x's initial value=2

$$s_0\,(L_0,2,0) \rightarrow s_1\,(L_1,1,0) \rightarrow s_2\,(L_0,1,0) \rightarrow s_3\,(L_1,0,0) \rightarrow s_4\,(L_0,0,0) \rightarrow s_5\,(L_2,0,0)$$

(note that $L_{i_0} = L_0$, $L_{i_1} = L_1$, $L_{i_2} = L_0$, , $L_{i_3} = L_1$, etc)

/* L_0 is executed infinitely, i.e., this program never halts */
L_0:x := x + 1
L_1:if y=0 then goto L_0 else y:=y-1
L_2:halt

Execution where x's initial value=0

$$s_0\,(L_0,0,0) \rightarrow s_1\,(L_1,1,0) \rightarrow s_2\,(L_0,1,0) \rightarrow s_3\,(L_1,2,0) \rightarrow s_4\,(L_0,2,0) \rightarrow \cdots\cdots$$

Validity in the predicate calculus

- Thm 5.43 (Church) Validity in the predicate calculus is undecidable
  - *Caution: the proof of Thm 5.43 in the textbook has several flaws…*
- For every two-register machine M, we construct a formula S_M s.t. S_M is valid iff M terminates when started in the state $(L_0,0,0)$:
  - $S_M = (\wedge_{(i=0..n-1)} S_i \wedge p_0(0,0)) \rightarrow \exists z_1 z_2\ p_n(z_1,z_2)$
    - Intuitive meaning of p_i is as follows
      - $v_{\mathcal{I}}(p_i(m',m'')) = T$ iff there exists some state $s_k=(L_i,m',m'')$
  - S_i is defined by cases of the instruction L_i

L_i	S_i
`x := x + 1`	$\forall x \forall y (p_i(x,y) \rightarrow p_{i+1}(s(x),y))$
`y := y + 1`	$\forall x \forall y (p_i(x,y) \rightarrow p_{i+1}(x,s(y)))$
`if x = 0` `then goto Lj` `else x := x - 1`	$\forall x (p_i(a,x) \rightarrow p_j(a,x)) \wedge$ $\forall x \forall y (p_i(s(x),y) \rightarrow p_{i+1}(x,y)))$
`if y = 0` `then goto Lj` `else y := y - 1`	$\forall x (p_i(x,a) \rightarrow p_j(x,a)) \wedge$ $\forall x \forall y (p_i(x,s(y)) \rightarrow p_{i+1}(x,y)))$

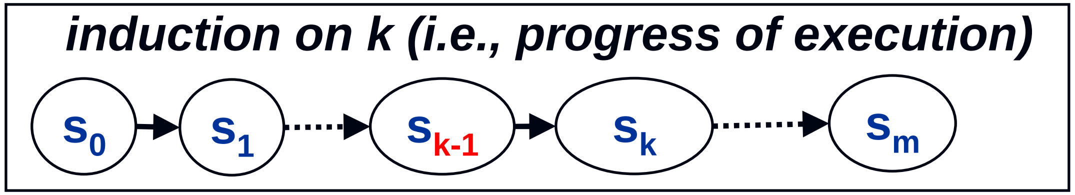
Example of S_M

/* y=x+1 for x <= 1* */ L_0:if x=0 then goto L_4 else x=x-1 L_1:y:=y+1 L_2:if x=0 then goto L_4 else x=x-1 L_3:y:=y+1 L_4:halt	$S_M = (p_0(0,0) \wedge$ $(\forall x(p_0(0,x) \rightarrow p_4(0,x)) \wedge \forall xy(p_0(s(x),y) \rightarrow p_1(x,y)))) \wedge$ $\forall xy\ (p_1(x,y) \rightarrow p_2(x,s(y)) \wedge$ $(\forall x(p_2(0,x) \rightarrow p_4(0,x)) \wedge \forall xy(p_2(s(x),y) \rightarrow p_3(x,y)))) \wedge$ $\forall xy(p_3(x,y) \rightarrow p_4(x,s(y))\ )$ $\rightarrow$ $\exists z_1 z_2\ p_4(z_1,z_2)$

- Intuitive meaning of S_M:
  - Given a two-register machine M,
    - execution of M ($\wedge_{i=0..n-1} S_i \wedge p_0(0,0)$)
    - reaches ($\rightarrow$)
    - the `halt` instruction ($\exists z_1 z_2\ p_n(z_1,z_2)$)

TRM halts $\rightarrow$ A_{TRM} is valid (1/2)

- Suppose that the execution $s_0,\ldots s_m$ of M halts and let $\mathcal{I}$ be an arbitrary interpretation for S_M. If $v_{\mathcal{I}}(S_i) = F$ (for $0 \leq i < n$) or $v_{\mathcal{I}}(p_0(0,0)) = F$, then trivially $v_{\mathcal{I}}(S_M) = T$
- Thus, we assume that $(\wedge_{(i=0..n-1)} S_i \wedge p_0(0,0))$ is true
  - since we need only consider interpretations that satisfy the antecedent of S_M
- We show by induction on k that $v_{\mathcal{I}}(\exists z_1 z_2\ p_{i_k}(z_1,z_2)) = T$
  - p_{i_k} is the predicate associated with the label L_{i_k} in state s_k
    - Mind the incorrect notation in the textbook where L_k and p_k is used instead of L_{i_k} and p_{i_k}
  - For k=0, $v_{\mathcal{I}}(\exists z_1 z_2\ p_{i_0}(z_1,z_2)) = v_{\mathcal{I}}(\exists z_1 z_2\ p_0(z_1,z_2)) = T$ since $v_{\mathcal{I}}(p_0(0,0))=T$ from the assumption



TRM halts $\rightarrow$ A_{TRM} is valid (2/2)

- For $k > 0$, the result follows by induction by cases according to the instruction at $L_{i_{k-1}}$
  - For $x := x+1$ at $L_{i_{k-1}}$:
    - $v_{\mathcal{I}}(\forall xy\ (p_{i_{k-1}}(x,y) \rightarrow p_{i_{k-1}+1}(s(x),y))) = T$ by the assumption
    - $v_{\mathcal{I}}(\exists z_1 z_2\ p_{i_{k-1}}(z_1,z_2)) = T$ by the inductive hypothesis
    - From the above two facts, $v_{\mathcal{I}}(\exists z_1 z_2\ p_{i_{k-1}+1}(s(z_1),z_2)) = T$
    - $v_{\mathcal{I}}(\exists z_1 z_2\ p_{i_{k-1}+1}(s(z_1),z_2)) = v_{\mathcal{I}}(\exists z_1 z_2\ p_{i_k}(s(z_1),z_2)) = T$ since $p_{i_{k-1}+1} = p_{i_k}$
      - We can conclude $v_{\mathcal{I}}(\exists z'_1 z_2\ p_{i_k}(z'_1,z_2)) = T$ since $\exists x\ A(f(x)) \rightarrow \exists x' A(x')$.
    - By induction, this holds for all k.
  - For if $x=0$ then goto L_j else $x=x-1$ at $L_{i_{k-1}}$:
    - …
    - By induction, this holds for all k.
- Since M halts, in the final state s_m, $L_{i_m} = L_n$ the halt instruction, so $v_{\mathcal{I}}(\exists z'_1 z_2\ p_n(z'_1,z_2)) = T$ and $v_{\mathcal{I}}(S_M) = T$.
- Since $\mathcal{I}$ was arbitrary, S_M is valid

TRM halts $\leftarrow$ A_{TRM} is valid

- Suppose that S_M is valid, and consider an interpretation $\mathcal{I}$ s.t.
  - $\mathcal{I} = (\mathcal{N}, \{P_0,\ldots,P_n\}, \{succ\}, \{0\})$ where
    - $(x,y) \in P_i$ iff (L_i,x,y) is reached by the register machine M when started in $(L_0,0,0)$
- We will show that the antecedent of S_M is true in $\mathcal{I}$ so that the conclusion of S_M is also true which means M reaches the halt instruction
  - The initial state is $(L_0,0,0)$ so $(0,0) \in P_0$ and $v_{\mathcal{I}}(p_0(0,0)) = T$
- We will show that if the computation has reached L_i, then $v_{\mathcal{I}}(S_i)=T$.
  - Assume as an inductive hypothesis that if the computation has reached L_i, it has done so in a computation of length–1 in state $s_{k-1}=(L_i,x_i,y_i)$, so $(x_i,y_i)\in P_i$.
  - The proof is by cases on the instruction L_i
    - For L_i = x:= x+1, the computation can reach the state $s_k = (L_{i_k}, succ(x_i), y_i) = (L_{i_{k-1}+1}, succ(x_i),y_i)$, so $v_{\mathcal{I}}(S_i) = T$
    - For L_i = if x=0 then goto L_j else x := x -1, …so $v_{\mathcal{I}}(S_i) = T$
- Since S_M is assumed valid, $v_{\mathcal{I}}(\exists z_1 z_2\ p_n(z_1,z_2))=T$ and $v_{\mathcal{I}}(p_n(m_1,m_2))=T$ for some natural numbers m_1, m_2. Thus M halts and computes $m_2 = f(0)$

Solvable cases

- Church's theorem holds even if the formulas contain only binary predicate symbols, one constant and one unary function symbol.
  - This follows from the structure of S_M in the proof
- Solvable cases of the decision problem
  - Thm 5.44 There is a decision procedure for validity of the class of pure formulas in Prenex CNF whose prefixes are of one of the following forms (where m,n $\geq$ 0)
    - $\forall x_1 \ldots x_n \exists y_1 \ldots y_m$ (Class $\forall^*\exists^*$)
    - $\forall x_1 \ldots x_n \exists y \forall z_1, \ldots z_m$ (Class $\forall^*\exists\forall^*$)
    - $\forall x_1 \ldots x_n \exists y_1 \exists y_2 \forall z_1 \ldots z_m$ (Class $\forall^*\exists\exists\forall^*$)
  - Def 7.8 A formula is in prenex conjuntive normal form (PCNF) iff it is of the form:
    - $Q_1x_1 \ldots Q_nx_n$ M where the Q_i are quantifiers and M is a quntifier-free formula in CNF. The sequence $Q_1x_1 \ldots Q_nx_n$ is called the prefix and M is called the matrix
  - Thm 5.46 There is a decision procedure for satisfiability of PCNF formulas A if the matrix of A satisfies the following
    - All atomic formulas are monadic, that is, all predicate letters are unary