

# Predicate Calculus

## - Natural deduction (2/2)

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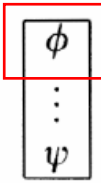
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<http://pswlab.kaist.ac.kr/courses/cs402-07>

	introduction	elimination
$\wedge$	$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\phi \wedge \psi}{\psi} \wedge e_2$
$\vee$	$\frac{\phi}{\phi \vee \psi} \vee i_1 \quad \frac{\psi}{\phi \vee \psi} \vee i_2$	$\frac{\phi \vee \psi \quad \begin{array}{ c } \hline \phi \\ \hline \vdots \\ \hline \chi \\ \hline \end{array} \quad \begin{array}{ c } \hline \psi \\ \hline \vdots \\ \hline \chi \\ \hline \end{array}}{\chi} \vee e$
$\rightarrow$	$\frac{\begin{array}{ c } \hline \phi \\ \hline \vdots \\ \hline \psi \\ \hline \end{array}}{\phi \rightarrow \psi} \rightarrow i$	$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e$
$\neg$	$\frac{\begin{array}{ c } \hline \phi \\ \hline \vdots \\ \hline \perp \\ \hline \end{array}}{\neg \phi} \neg i$	$\frac{\phi \quad \neg \phi}{\perp} \neg e$
$\perp$	(no introduction rule for $\perp$ )	$\frac{\perp}{\phi} \perp e$
$\neg\neg$		$\frac{\neg\neg\phi}{\phi} \neg\neg e$

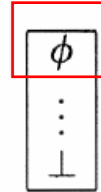
assumption



assumption



assumption

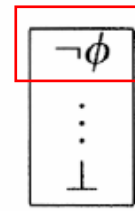


# Summary of proof rules of natural deduction

$$\frac{\phi \rightarrow \psi \quad \neg \psi}{\neg \phi} \text{MT}$$

$$\frac{\phi}{\neg \neg \phi} \neg\neg i$$

assumption



$$\frac{}{\phi} \text{RAA}$$

$$\frac{}{\phi \vee \neg \phi} \text{LEM}$$

$$\frac{\begin{array}{|c|} \hline x_0 \\ \hline \vdots \\ \hline \phi[x_0/x] \\ \hline \end{array}}{\forall x \phi} \forall x i$$

$$\frac{\forall x \phi}{\phi[t/x]} \forall x e$$

$$\frac{\phi[t/x]}{\exists x \phi} \exists x i$$

$$\frac{\exists x \phi \quad \begin{array}{|c|} \hline x_0 \quad \phi[x_0/x] \\ \hline \vdots \\ \hline \chi \\ \hline \end{array}}{\chi} \exists x e$$

# Example

$\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$

- |   |                                     |                     |
|---|-------------------------------------|---------------------|
| ① | $\forall x (P(x) \rightarrow Q(x))$ | Premise             |
| ② | $\exists x P(x)$                    | Premise             |
| ③ | $x_0 P(x_0)$                        | Assumption          |
| ④ | $P(x_0) \rightarrow Q(x_0)$         | $\forall x e 1$     |
| ⑤ | $Q(x_0)$                            | $\rightarrow e 4,3$ |
| ⑥ | $\exists x Q(x)$                    | $\exists x i 5$     |
| ⑦ | $\exists x Q(x)$                    | $\exists x e 2,3-6$ |

$\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$

- |   |                                     |                     |
|---|-------------------------------------|---------------------|
| ① | $\forall x (P(x) \rightarrow Q(x))$ | Premise             |
| ② | $\exists x P(x)$                    | Premise             |
| ③ | $x_0 P(x_0)$                        | Assumption          |
| ④ | $P(x_0) \rightarrow Q(x_0)$         | $\forall x e 1$     |
| ⑤ | $Q(x_0)$                            | $\rightarrow e 4,3$ |
| ⑥ | $Q(x_0)$                            | $\exists x e 2,3-5$ |
| ⑦ | $\exists x Q(x)$                    | $\exists x i 6$     |

What is wrong with this proof?

$$\forall x(Q(x) \rightarrow R(x)), \exists x(P(x) \wedge Q(x)) \vdash \exists x(P(x) \wedge R(x))$$

- This formula may read as follows:
  - If all quakers ( $Q(x)$ ) are reformists ( $R(x)$ ) and if there is a protestant ( $P(x)$ ) who is also a quaker, **then** there must be a protestant who is also a reformist

①  $\forall x(Q(x) \rightarrow R(x))$  premise

②  $\exists x(P(x) \wedge Q(x))$  premise

③

④

⑤

⑥

⑦

⑧

⑨

⑩  $\exists x (P(x) \wedge R(x))$

$$\exists x P(x), \forall xy (P(x) \rightarrow Q(y)) \vdash \forall y Q(y)$$

- |   |   |                     |
|---|---|---------------------|
| ① | $\exists x P(x)$                              | premise             |
| ② | $\forall x \forall y (P(x) \rightarrow Q(x))$ | premise             |
| ③ | $y_0$   |                     |
| ④ | $x_0 P(x_0)$                                  | assumption          |
| ⑤ | $\forall y P(x_0) \rightarrow Q(y)$           | $\forall x$ e 2     |
| ⑥ | $P(x_0) \rightarrow Q(y_0)$                   | $\forall y$ e 5     |
| ⑦ | $Q(y_0)$                                      | $\rightarrow$ e 6,4 |
| ⑧ | $Q(y_0)$                                      | $\exists x$ e 1,4-7 |
| ⑨ | $\forall y Q(y)$                              | $\forall y$ i 3-8   |

$\exists x P(x), \forall x (P(x) \rightarrow Q(x)) \vdash \forall y Q(y) ???$

①	$\exists x P(x)$	premise
②	$\forall x (P(x) \rightarrow Q(x))$	premise
③	$x_0$	
④	$x_0 P(x_0)$	assumption
⑤	$P(x_0) \rightarrow Q(x_0)$	$\forall x e 2$
⑥	$Q(x_0)$	$\rightarrow e 5,4$
⑦	$Q(x_0)$	$\exists x e 1,4-6$
⑧	$\forall y Q(y)$	$\forall y i 3-7$

**/\* Compare with**  
 $\exists x P(x), \forall xy (P(x) \rightarrow Q(y))$   
 $\vdash \forall y Q(y)$   
**\*/**

$$\neg \forall x \phi \vdash \exists x \neg \phi$$

1	$\neg \forall x P(x)$		premise
2	$\neg \exists x \neg P(x)$		assumption
3	$x_0$		
4	$\neg P(x_0)$		assumption
5	$\exists x \neg P(x)$		$\exists x$ i 4
6	$\perp$		$\neg$ e 5, 2
7	$P(x_0)$		RAA 4–6
8	$\forall x P(x)$		$\forall x$ i 3–7
9	$\perp$		$\neg$ e 8, 1
10	$\exists x \neg P(x)$		RAA 2–9

1	$\neg \forall x \phi$		premise
2	$\neg \exists x \neg \phi$		assumption
3	$x_0$		
4	$\neg \phi[x_0/x]$		assumption
5	$\exists x \neg \phi$		$\exists x$ i 4
6	$\perp$		$\neg$ e 5, 2
7	$\phi[x_0/x]$		RAA 4–6
8	$\forall x \phi$		$\forall x$ i 3–7
9	$\perp$		$\neg$ e 8, 1
10	$\exists x \neg \phi$		RAA 2–9

$$\neg \forall x \phi \vdash \exists x \neg \phi$$

1	$\exists x \neg \phi$	assumption
2	$\forall x \phi$	assumption
3	$x_0$	
4	$\neg \phi[x_0/x]$	assumption
5	$\phi[x_0/x]$	$\forall x e 2$
6	$\perp$	$\neg e 5, 4$
7	$\perp$	$\exists x e 1, 3-6$
8	$\neg \forall x \phi$	$\neg i 2-7$



# Comparison between Semantic tableau

$$\forall x (p(x) \rightarrow q(x)) \vdash (\forall x p(x) \rightarrow \forall x q(x))$$

■  $\forall x (p(x) \rightarrow q(x)) \vdash (\forall x p(x) \rightarrow \forall x q(x))$  ■

■  $\vdash \forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$

$$\neg (\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x)))$$

$$\forall x (p(x) \rightarrow q(x)), \neg (\forall x p(x) \rightarrow \forall x q(x))$$

$$\forall x (p(x) \rightarrow q(x)), \forall x p(x), \neg \forall x q(x)$$

$$\forall x (p(x) \rightarrow q(x)), \forall x p(x), \neg q(a)$$

$$\forall x (p(x) \rightarrow q(x)), p(b), \neg q(a)$$

$$p(b) \rightarrow q(a), p(b), \neg q(a)$$

$$\begin{array}{l} \neg p(b), p(b), \neg q(a) \quad q(a), p(b), \neg q(b) \\ \mathbf{x} \qquad \qquad \qquad \mathbf{x} \end{array}$$

$$\forall x (p(x) \rightarrow q(x)) \vdash (\forall x p(x) \rightarrow \forall x q(x))$$

①  $\forall x (p(x) \rightarrow q(x))$  Premise

②  $\forall x (p(x))$  assumption

③  $x_0$   $p(x_0) \rightarrow q(x_0)$   $\forall x$  e 1

④  $p(x_0)$   $\forall x$  e 2

⑤  $q(x_0)$   $\rightarrow$  e 2,3

⑥  $\forall x q(x)$   $\forall x$  i

⑦  $\forall x p(x) \rightarrow \forall x q(x)$   $\rightarrow$  i 2-6