Propositional Calculus
- *Natural deduction*

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Review

- Goal of logic
  - To check whether given a formula ϕ is valid
  - To prove a given formula ϕ

Sound & Complete

$\vdash \phi \quad \equiv \quad \models \phi$

Syntactic method (proof system $\mathcal{G}$, $\mathcal{H}$, natural deduction,etc)

Semantic method (truth table, etc)

Sound & Complete $\equiv$ **Semantic tableau** $\equiv$ ***Sound & Complete***

Natural deduction

- A variant of Gentzen system
- In natural deduction, similar to other deductive proof systems such as $\mathcal{G}$ and $\mathcal{H}$, we have a collection of proof rules.
- But natural deduction does not have axioms.

Proof rules (1/3)

	introduction	*elimination*	
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1$	$\dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$

- $\wedge$i says: to prove $\phi \wedge \psi$, you must first prove ϕ and ψ separately and then use the rule $\wedge$ i.
- $\wedge$ e_1 says: to prove ϕ, try proving $\phi \wedge \psi$ and then use the rule $\wedge$ e_1. Actually this does not sound like very good advice because probably proving $\phi \wedge \psi$ will be harder than proving ϕ alone. However, you might find that you already have $\phi \wedge \psi$ lying around, so that's when this rule is useful.

Proof rules (1/3)

	introduction	elimination
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee i_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee i_2$	$\dfrac{\phi \vee \psi \quad \overset{\text{assumption 1}}{\boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}}} \quad \overset{\text{assumption 2}}{\boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}}{\chi}\ \vee e$

- $\vee\, i_1$ says: to prove $\phi \vee \psi$, try proving ϕ. Again, in general it is harder to prove ϕ than it is to prove $\phi \vee \psi$, so this will usually be useful only if you have already managed to prove ϕ.
- $\vee\, e$ has an excellent procedural interpretation. It says: if you have $\phi \vee \psi$, and you want to prove some χ, then try to prove χ from ϕ and from ψ in turn
  - In those subproofs, of course you can use the other prevailing premises as well

Proof rules (3/3)

	introduction	*elimination*
$\rightarrow$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \psi\end{matrix}}}{\phi \rightarrow \psi} \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \bot\end{matrix}}}{\neg\phi} \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot} \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi} \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi} \neg\neg\mathrm{e}$

Some useful derived rules

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad\qquad \frac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

- At any stage of a proof, it is permitted to introduce any formula as assumption, by choosing a proof rule that opens a box. As we saw, natural deduction employs boxes to control the scope of assumptions.
- When an assumption is introduced, a box is opened. Discharging assumptions is achieved by closing a box according to the pattern of its particular proof rule.

Example 1

$$p \wedge \neg q \rightarrow r,\ \neg r,\ p \vdash q$$

1	$p \wedge \neg q \rightarrow r$	premise
2	$\neg r$	premise
3	p	premise
4	$\neg q$	assumption
5	$p \wedge \neg q$	$\wedge$i 3, 4
6	r	$\rightarrow$e 1, 5
7	$\bot$	$\neg$e 6, 2
8	$\neg\neg q$	$\neg$i 4–7
9	q	$\neg\neg$e 8

Example 2

$$p \to q \vdash \neg p \lor q$$

1	$p \to q$	premise
2	$\neg p \lor p$	LEM
3	$\neg p$	assumption
4	$\neg p \lor q$	$\lor i_1$ 3
5	p	assumption
6	q	$\to$e 1, 5
7	$\neg p \lor q$	$\lor i_2$ 6
8	$\neg p \lor q$	$\lor$e 2, 3–4, 5–7

Example 3 (Law of Excluded Middle)

$$\frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee i_1$ 2
4	$\perp$	$\neg e$ 3, 1
5	$\neg\phi$	$\neg i$ 2–4
6	$\phi \vee \neg\phi$	$\vee i_2$ 5
7	$\perp$	$\neg e$ 6, 1
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg i$ 1–7
9	$\phi \vee \neg\phi$	$\neg\neg e$ 8

Summary of proof rules

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

$$\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \dfrac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad\qquad \dfrac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$