

1. (1) $\forall x \forall y \forall z (S(x, y) \wedge S(y, z) \rightarrow S(x, z)), \forall x S(x, x) \vdash \forall x \forall y (S(x, y) \rightarrow \neg S(y, x))$

1. $\forall x \forall y \forall z (S(x, y) \wedge S(y, z) \rightarrow S(x, z))$

premise.

2. $\forall x \neg S(x, x)$

premise.

3. x_0

4. y_0

5. $S(x_0, y_0)$

assumption.

6. $S(y_0, x_0)$

assumption

7. $S(x_0, y_0) \wedge S(y_0, x_0)$

$\wedge i$ 5, 6

8. $\forall y \forall z (S(x_0, y) \wedge S(y, z) \rightarrow S(x_0, z))$

$\forall x e$, 1.

9. $\forall z (S(x_0, y_0) \wedge S(y_0, z) \rightarrow S(x_0, z))$

$\forall y e$, 8

10. $S(x_0, y_0) \wedge S(y_0, x_0) \rightarrow S(x_0, x_0)$

$\forall z e$, 9

11. $S(x_0, x_0)$

$\rightarrow e$, 7, 10

12. $\neg S(x_0, x_0)$

$\forall x e$, 2.

13. \perp

$\neg e$, 11, 12

14. $\neg S(y_0, x_0)$

$\neg i$, 6 ~ 13.

15. $S(x_0, y_0) \rightarrow \neg S(y_0, x_0)$

$\rightarrow i$ 5 ~ 14.

16. $\forall y (S(x_0, y) \rightarrow \neg S(y, x_0))$

$\forall y i$, 4 ~ 15

17. $\forall x \forall y (S(x, y) \rightarrow \neg S(y, x))$

$\forall x i$, 3 ~ 16

(b) $\forall x (P(x) \vee Q(x)), \exists x \neg Q(x), \forall x (R(x) \rightarrow \neg P(x)) \vdash \exists x \neg R(x)$

1. $\forall x (P(x) \vee Q(x))$ premise

2. $\exists x \neg Q(x)$ premise

3. $\forall x (R(x) \rightarrow \neg P(x))$ premise

4. $x_0, \neg Q(x_0)$ assumption

5. $P(x_0) \vee Q(x_0)$ $\forall x e, 1.$

6. $\neg P(x_0)$ assumption

7. $P(x_0)$ assumption

8. \perp $\neg e, 6, 7$

9. $Q(x_0)$ assumption

10. \perp $\neg e, 9, 4$

11. \perp $\forall e, 5, 7 \sim 8, 9 \sim 10$

12. $\neg P(x_0)$ $\neg i, 6 \sim 11$

13. $R(x_0) \rightarrow \neg P(x_0)$ $\forall x e, 3.$

14. $\neg R(x_0)$ MT 13, 12

15. $\exists x \neg R(x)$ $\exists x i, 14$

16. $\exists x \neg R(x)$ $\exists x e, 2, 4 \sim 15.$

(c) $\forall x (P(x) \rightarrow (Q(x) \vee R(x))), \neg \exists x (P(x) \wedge R(x)) \vdash \forall x (P(x) \rightarrow Q(x))$

1. $\forall x (P(x) \rightarrow (Q(x) \vee R(x)))$ premise

2. $\neg \exists x (P(x) \wedge R(x))$ premise

3. x_0

4. $P(x_0) \rightarrow (Q(x_0) \vee R(x_0))$ $\forall x e, 1$

5. $P(x_0)$ assumption

6. $Q(x_0) \vee R(x_0)$ $\rightarrow e, 4, 5$

7. $\neg Q(x_0)$ assumption

8. $\neg R(x_0)$ assumption

9. $Q(x_0)$ assumption

10. \perp $\neg e, 7, 9$

11. $R(x_0)$ assumption

12. \perp $\neg e, 8, 11$

13. \perp $\vee e, 5, 9 \sim 10, 11 \sim 12$

14. $R(x_0)$ RAA, $8 \sim 13$

15. $P(x_0) \wedge R(x_0)$ $\wedge i, 5, 14$

16. $\exists x (P(x) \wedge R(x))$ $\exists x i, 15$

17. \perp $\neg e, 2, 16$

18. $Q(x_0)$ RAA $9 \sim 17$

19. $P(x_0) \rightarrow Q(x_0)$ $\rightarrow i, 5 \sim 18$

20. $\forall x (P(x) \rightarrow Q(x))$ $\forall x i, 4 \sim 19$

(d) $\exists x \exists y (S(x,y) \vee S(y,x)) \vdash \exists x \exists y S(x,y)$

1. $\exists x \exists y (S(x,y) \vee S(y,x))$

premise

2. $x_0 \exists y (S(x_0,y) \vee S(y,x_0))$

assumption

3. $y_0 S(x_0,y_0) \vee S(y_0,x_0)$

assumption

4. $S(x_0,y_0)$

assumption

5. $\exists y S(x_0,y)$

$\exists y_i, 4$

6. $\exists x \exists y S(x,y)$

$\exists x_i, 5$

7. $S(y_0,x_0)$

assumption

8. $\exists y S(y,x_0)$

$\exists y_i, 7$

9. $\exists x \exists y S(x,y)$

$\exists x_i, 8$

10. $\exists x \exists y S(x,y)$

$\forall e, 4, 6, 9$

11. $\exists x \exists y S(x,y)$

$\exists y e, 2, 3 \sim 10$

12. $\exists x \exists y S(x,y)$

$\exists x e, 1, 2 \sim 11$

(e) $\exists x (P(x) \wedge Q(x)), \forall x (P(x) \rightarrow R(x)) \vdash \exists x (R(x) \wedge Q(x))$

1. $\exists x (P(x) \wedge Q(x))$

premise.

2. $\forall x (P(x) \rightarrow R(x))$

premise

3. $x_0 P(x_0) \wedge Q(x_0)$

assumption

4. $P(x_0)$

$\wedge e, 3$

5. $Q(x_0)$

$\wedge e, 3$

6. $P(x_0) \rightarrow R(x_0)$

$\forall e, 2$

7. $R(x_0)$

$\rightarrow e, 4, 6$

8. $R(x_0) \wedge Q(x_0)$

$\wedge i, 7, 5$

9. $\exists x (R(x) \wedge Q(x))$

$\exists x i, 8$

10. $\exists x (R(x) \wedge Q(x))$

$\exists x e, 1, 3 \sim 9$

2. a) $\neg \exists x Q(x) \vdash \exists x (\neg Q(x))$

1 $\neg \exists x Q(x)$ premise

2 $\neg \neg \neg$ LEM

3 \neg assumption

4 $\exists x Q(x)$ \rightarrow e, 1, 4

5 $\neg_0 Q(x_0)$ assumption

6 \neg assumption

7 $\neg Q(x_0)$ assumption

8 \perp \neg e, 5, 7

9 $Q(x_0)$ \perp e, 8

10 $\neg Q(x_0)$ \rightarrow i, 6, 9

11 $\exists x (\neg Q(x))$ \exists i, 10

12 $\exists x (\neg Q(x))$ \exists e, 4, 5-11

13 $\neg \neg$ assumption

14 \neg assumption

15 \perp \neg e, 13, 14

16 $Q(x_0)$ \perp e, 15

17 $\neg Q(x_0)$ \rightarrow i, 14-16

18 $\exists x (\neg Q(x))$ \exists i, 17

19 $\exists x (\neg Q(x))$ \forall e, 3-12, 13-18.

2. (b) $\exists x p(x) \rightarrow s \vdash \forall x (p(x) \rightarrow s)$

1.	$\exists x p(x) \rightarrow s$			premise
2.	x_0			
3.		$p(x_0)$		assumption
4.		$\exists x p(x)$		$\exists x i, 3$
5.		s		$\rightarrow e, 1, 4$
6.		$p(x_0) \rightarrow s$		$\rightarrow i, 3 \sim 5$
7.	$\forall x (p(x) \rightarrow s)$			$\forall x i, 6$

2. (c) $\forall x p(x) \rightarrow s \vdash \exists x (p(x) \rightarrow s)$

1.	$\forall x p(x) \rightarrow s$			premise
2.	$\forall x p(x) \vee \neg \forall x p(x)$			LEM
3.	$\forall x p(x)$			assumption
4.		$p(x_0)$		assumption (x_0 is not a fresh variable)
5.		s		$\rightarrow e, 1, 3$
6.		$p(x_0) \rightarrow s$		$\rightarrow i, 4 \sim 5$
7.	$\exists x (p(x) \rightarrow s)$			$\exists x i, 6$

8	$\neg \forall x P(x)$		assumption.
9	$\exists x \neg P(x)$		assumption.
10	x_1		
11	$\neg P(x_1)$		assumption
12	$\exists x \neg P(x)$		$\exists x_i 11$
13	\perp		$\neg e, 9, 12$
14	$P(x_1)$		RAA 11~13
15	$\forall x P(x)$		$\forall x_i 10 \sim 14$
16	\perp		$\neg e, 8, 15$
17	$\exists x \neg P(x)$		RAA 9~16
18	$x_2 \neg P(x_2)$		assumption.
19	$P(x_2)$		assumption.
20	\perp		$\neg e, 18, 19$
21	S		$\perp e, 20$
22	$P(x_2) \rightarrow S$		$\rightarrow i 18 \sim 21$
23	$\exists x (P(x) \rightarrow S)$		$\exists x_i 22$
24	$\exists x (P(x) \rightarrow S)$		$\exists x e, 17, 18 \sim 23$
25.	$\exists x (P(x) \rightarrow S)$		$\forall e, 2, 3 \sim 7, 8 \sim 24$

	2. (d) $\forall x (P(x) \vee Q(x)) \vdash \forall x P(x) \vee \exists x Q(x)$		
1	$\forall x (P(x) \vee Q(x))$		premise
2	$\forall x P(x) \vee \neg \forall x P(x)$		LEM
3	$\forall x P(x)$		assumption
4	$\forall x P(x) \vee \exists x Q(x)$		$\vee i, 3$
5	$\neg \forall x P(x)$		assumption
6	$\exists x \neg P(x)$		quantifier equivalence, proof in slides
7	$x_0 \neg P(x_0)$		assumption
8	$P(x_0) \vee Q(x_0)$		$\forall x e, 1$
9	$P(x_0)$		assumption
10	\perp		$\neg e, 7, 9$
11	$Q(x_0)$		$\perp e, 10$
12	$Q(x)$		assumption
13	$Q(x_0)$		$\vee e, 8, 9 \sim 11, 12$
14	$\exists x Q(x)$		$\exists x i, 13$
15	$\exists x Q(x)$		$\exists x e, 6, 14 \sim 14$
16	$\forall x P(x) \vee \exists x Q(x)$		$\vee i, 15$
17	$\forall x P(x) \vee \exists x Q(x)$		$\vee e, 2, 3 \sim 4, 5 \sim 16$

	$2. (e) \forall x \exists y (p(x) \vee q(y)) \vdash \exists y \forall x (p(x) \vee q(y))$		
1.	$\forall x \exists y (p(x) \vee q(y))$		premise
2.	$\exists y q(y) \vee \neg \exists y q(y)$		LEM
3.	$\exists y q(y)$		assumption
4.	z_0	$q(z_0)$	assumption
5.		z_0	
6.		$p(z_0) \vee q(z_0)$	$\vee i 2, 4$
7.		$\forall x (p(x) \vee q(z_0))$	$\forall x i 5 \sim 6$
8.		$\exists y \forall x (p(x) \vee q(y))$	$\exists y i 7$
9.		$\exists y \forall x (p(x) \vee q(y))$	$\exists y e 3, 4 \sim 8$

