Predicate Calculus
- Semantics 3/4

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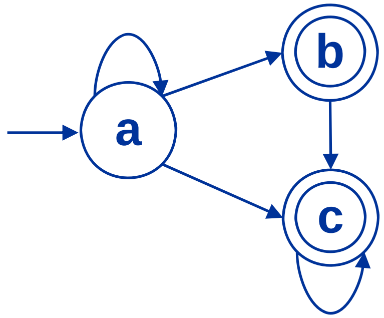
Example: finite automata

- For an interpretation $\mathcal{I} = (\mathcal{D},\mathcal{R},\mathcal{F},\mathcal{C})$ where
  - $\mathcal{D}$ = {a,b,c}
  - $\mathcal{R}$= {Trans, Final, Equality} where
    - Trans = {(a,a),(a,b),(a,c),(b,c),(c,c)}
    - Final = {b,c}
    - Equality={(a,a),(b,b),(c,c)}
  - $\mathcal{F}$={}
  - $\mathcal{C}$={a}



- Formulas for $\mathcal{I}$ where $R^{\mathcal{I}}$=Trans, $F^{\mathcal{I}}$=Final, $=^{\mathcal{I}}$ =Equality, $i^{\mathcal{I}}$=a
  - $\mathcal{I} \models \exists y\, R(i,y)$
  - $\mathcal{I} \models \neg F(i)$
  - $\mathcal{I} \not\models \forall x \forall y \forall z\, (R(x,y) \wedge R(x,z) \rightarrow y = z)$
  - $\mathcal{I} \models \forall x \exists y\, R(x,y)$

A formula represents a set of models

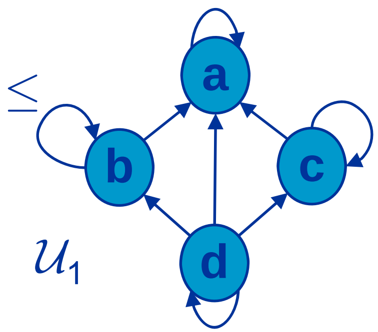
- A formula ϕ describes characteristics of target structures in a compact way.
  - ex. deterministic automata, partial order sets, binary trees, relational database, etc
- In other words, a formula ϕ designates a set of models (i.e., interpretations) that satisfies ϕ
  - $\forall x \forall y \forall z\ (R(x,y) \wedge R(x,z) \rightarrow y = z)$ represents all deterministic graphs
  - $\forall x \forall y \forall z\ (R(x,y) \wedge R(y,z) \rightarrow R(x,z))$ represents all transitive graphs.
- Validity, satisfiability, and provability of a predicate formula is all undecidable. However, checking formulas on concrete interpretations is practical
  - ex. SQL queries over relational database
  - ex. XQueries over XML documents
  - ex. Model checking of a program

Example: partial order set (POSET)

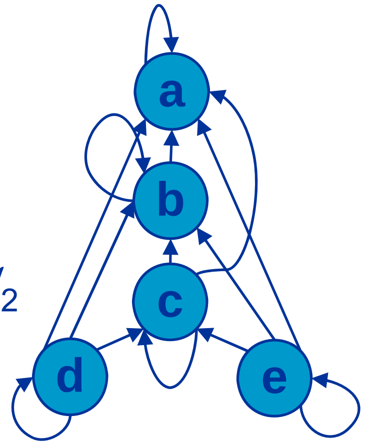
- Def. $\mathcal{U}$ is a partially ordered set (poset) if $\mathcal{U}$ is a model of
  - $\forall xyz\ (p(x,y) \land p(y,z) \rightarrow p(x,z))$
  - $\forall xy\ (p(x,y) \land p(y,x) \leftrightarrow q(x,y))$

  $p^{\mathcal{U}} = \leq$, $q^{\mathcal{U}} = =$, then
  - $\forall xyz\ (x \leq y \leq z \rightarrow x \leq z)$
  - $\forall xy\ (x \leq y \leq x \leftrightarrow x = y)$
- $\mathcal{U}_1 \models \exists x\ \forall y\ (x \leq y)$
  - i.e., $\mathcal{U}_1$ has a least element
- $\mathcal{U}_3 \models \forall x \neg \exists y\ (x < y)$
  - i.e., in $\mathcal{U}_3$ no element is strictly less than another element

- Note that $x<y \equiv x \leq y \land \neg(x=y)$
- Def. $\mathcal{U}$ is a totally ordered set if $\mathcal{U}$ is a poset and $\mathcal{U} \models \forall x\ \forall y\ (x \leq y \lor y \leq x)$
- Def. $\mathcal{U}$ is densely ordered if $\mathcal{U} \models \forall x\ \forall y\ (x < y \rightarrow \exists z\ (x<z \land z< y))$
- We can distinguish $\mathcal{U}_3$ and $\mathcal{U}_4$ by $A(x) = \forall y\ (y \neq x \rightarrow \neg(y \leq x) \land \neg(x \leq y))$
  - $\mathcal{U}_4 \models \forall x\ \forall y\ (A(x) \land A(y) \rightarrow x = y)$
  - $\mathcal{U}_3 \models \neg\forall x\ \forall y\ (A(x) \land A(y) \rightarrow x = y)$

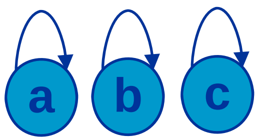


$\mathcal{U}_1$

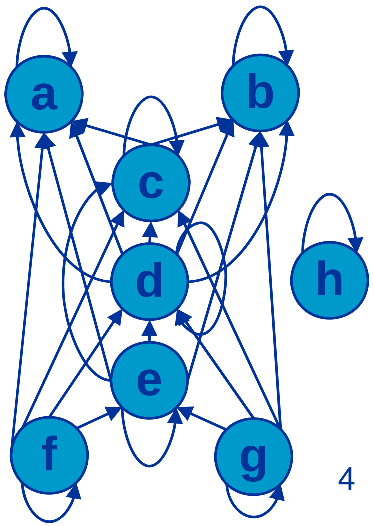


$\mathcal{U}_2$

$\mathcal{U}_3$



$\mathcal{U}_4$



Exercise: POSET (cont.)

- Define formulas for
  - x is the maximum (the largest element in a target domain)
    - $\forall y\ y \leq x$
  - x is maximal (not smaller than any other elements)
    - $\neg\exists y\ x < y \equiv \forall y\ \neg(x < y)$
    - Note the difference between $\forall y\ y \leq x$ and $\forall y\ \neg(x < y)$.
      - For totally ordered set, these two formulas are same, but for POSET, they are different.
  - There is no element between x and y
    - $\neg\exists z\ ((x \leq z \wedge z \leq y) \vee (y \leq z \wedge z \leq x))$
  - x is an immediate successor of y
    - $(x > y) \wedge \neg\exists z\ (y \leq z \wedge z \leq x)$
  - z is the infimum of x and y (the greatest element less than or equal to x and y)
    - $\forall st\ ((s\leq x \wedge t\leq y) \rightarrow (s\leq z \wedge t\leq z) \wedge (z\leq x \wedge z\leq y))$
- Give a formula ϕ s.t. $\mathcal{U}_2 \vDash \phi$ and $\mathcal{U}_4 \vDash \neg\phi$
- Let $\phi = \exists x\forall y\ (x \leq y \vee y \leq x)$. Find posets $\mathcal{U}_1$ and $\mathcal{U}_2$ s.t. $\mathcal{U}_1 \vDash \phi$ and $\mathcal{U}_2 \vDash \neg\ \phi$

Example: arithmetic

- Def. A Peano structure $\mathcal{U} = (\mathcal{N}$, {=}, {S,+,*}, {0}) is a model of
  1. $\forall x\ (\neg\ (0 = S(x)))$
  2. $\forall x\ \forall y\ (S(x) = S(y) \rightarrow x = y)$
  3. $\forall x\ (x + 0 = x)$
  4. $\forall x\ \forall y\ (x+S(y) = S(x+y))$
  5. $\forall\ x\ (x * 0 = 0)$
  6. $\forall x\ \forall y\ (x*S(y) = x*y + x)$
  7. $\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(S(x)) \rightarrow \forall x \phi(x)$
     - mathematical induction
- These 7 formulas do not have "$\leq$" or "<" but these predicate can be expressed by
  - $x < y ::= \exists\ z\ (x + S(z) = y)$
  - $x \leq y ::= x < y \vee x = y$

- Example
  - The set of even numbers
    - $E(x) ::= \exists\ y\ (x = y + y)$
  - The divisibility relation
    - $x|y ::= \exists\ z\ (x*z = y)$
  - The set of prime numbers
    - $P(x) ::=$

    $\forall y \forall z\ (x = y*z \rightarrow y=1 \vee z=1) \wedge x \neq 1$