Propositional Calculus
- Natural deduction

Moonzoo Kim
CS Dept. KAIST

moonzoo@cs.kaist.ac.kr

Review

- Goal of logic
  - To check whether given a formula ϕ is valid
  - To prove a given formula ϕ

Sound & Complete

$\vdash \phi$ $\equiv$ $\models \phi$

Syntactic method (proof system $\mathcal{G}$, $\mathcal{H}$, natural deduction,etc)

Semantic method (truth table, etc)

$\equiv$ **Sound & Complete**

$\equiv$ **Sound & Complete**

Semantic tableau

Deductive proofs (1/3)

- Suppose we want to know if ϕ belongs to the theory $\mathcal{T}$(U).
  - By Thm 2.38 U $\models \phi$ iff $\models A_1 \wedge \ldots \wedge A_n \rightarrow \phi$ where U = { $A_1,\ldots,A_n$}
  - Thus, $\phi \in \mathcal{T}$(U) iff a decision procedure for validity answers 'yes'
- However, there are several problems with this semantic approach
  - The set of axioms may be infinite
    - e.x. Hilbert deductive system $\mathcal{H}$ has an axiom schema (A $\rightarrow$ (B $\rightarrow$ A)), which generates an infinite number of axioms by replacing schemata variables A,B and C with infintely many subformulas (e.g. $\phi\wedge\psi$,$\neg\phi\vee\psi$, etc)
    - e.x.2. Peano and ZFC theories cannot be finitely axiomatized.
  - Very few logics have decision procedures for validity of ϕ
    - ex. propositional logic has a decision procedure using truth table
    - ex2. predicate logic does not have such decision procedure
- There is another approach to logic called deductive proofs.
  - Instead of working with semantic concepts like interpretation/model and consequence
  - we choose a set of axioms and a set of syntactical rules for deducing new formulas from the axioms

Deductive proofs (2/3)

- Def 3.1
  - A deductive system consists of
    - a set of axioms and
    - a set of inference rules
  - A proof in a deductive system is a sequence of sets of formulas s.t. each element is either an axiom or it can be inferred from previous elements of the sequence using a rule of inference
  - If {A} is the last element of the sequence, A is a theorem, the sequence is a proof of A, and A is provable, denoted $\vdash$ A
- Example of a proof of $(p\vee q)\rightarrow(q\vee p)$ in gentzen system $\mathcal{G}$
  - $\{\neg p,q,p\}.\{\neg q,q,p\}.\{\neg(p\vee q),q,p\}.\{\neg(p\vee q),(q\vee p)\}.\{(p\vee q)\rightarrow(q\vee p)\}$

axioms

theorem

- tree representation of this proof is more intuitive

$$\boxed{\neg p, q, p} \quad \searrow \ \beta\vee \ \swarrow \quad \boxed{\neg q, q, p}$$

$$\neg(p\vee q), q, p$$

$$\downarrow \alpha\vee$$

$$\neg(p\vee q), (q\vee p)$$

$$\downarrow \alpha\rightarrow$$

$$(p\vee q)\rightarrow(q\vee p)$$

Deductive proofs (3/3)

- Deductive proofs has following benefits
  - There may be an infinite number of axioms, but only a finite number of axioms will appear in any proof
  - Any particular proof consists of a finite sequence of sets of formulas, and the legality of each individual deduction can be easily and efficiently determined from the syntax of the formulas
  - The proof of a formula clearly shows which axioms, theorems and rules are used and for what purposes.
    - Such a pattern (i.e. relationship between formulas) can then be transferred to other similar proofs, or modified to prove different results.
    - Lemmas and corollaries can be obtained and can be used later
- But with a new problem
  - deduction is defined purely in terms of syntactical formula manipulation
  - But it is not amenable to systematic search procedures
    - no brute-force search is possible because any axiom can be used

Natural deduction

- In natural deduction, similar to other deductive proof systems such as $\mathcal{G}$ and $\mathcal{H}$, we have a collection of proof rules.
  - Natural deduction does not have axioms.
- Suppose we have premises .and $\phi_1, \phi_2, \ldots, \phi_n$ and would like to prove a conclusion ψ. The intention is denoted by

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

  We call this expression a sequent; it is valid if a proof for it can be found
- Def: A logical formula ϕ with valid sequent $\vdash \phi$ is theorem

Proof rules (1/3)

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi} \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi} \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi} \wedge\mathrm{e}_2$

- $\wedge$i says: to prove $\phi \wedge \psi$, you must first prove ϕ and ψ separately and then use the rule $\wedge$ i.
- $\wedge$ e_1 says: to prove ϕ, try proving $\phi \wedge \psi$ and then use the rule $\wedge$ e_1. Actually this does not sound like very good advice because probably proving $\phi \wedge \psi$ will be harder than proving ϕ alone. However, you might find that you already have $\phi \wedge \psi$ lying around, so that's when this rule is useful.

Proof rules (1/3)

	introduction	*elimination*
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee i_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee i_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi}\ \vee e$

assumption 1

assumption 2

- $\vee\, i_1$ says: to prove $\phi \vee \psi$, try proving ϕ. Again, in general it is harder to prove ϕ than it is to prove $\phi \vee \psi$, so this will usually be useful only if you have already managed to prove ϕ.
- $\vee$ e has an excellent procedural interpretation. It says: if you have $\phi \vee \psi$, and you want to prove some χ, then try to prove χ from ϕ and from ψ in turn
  - In those subproofs, of course you can use the other prevailing premises as well

Proof rules (3/3)

	introduction	*elimination*
$\to$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}\ {\to}\mathrm{i}$	$\dfrac{\phi \quad \phi \to \psi}{\psi}\ {\to}\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

Some useful derived rules

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad\qquad \frac{}{\phi \lor \neg\phi}\ \mathrm{LEM}$$

- At any stage of a proof, it is permitted to introduce any formula as assumption, by choosing a proof rule that opens a box. As we saw, natural deduction employs boxes to control the scope of assumptions.
- When an assumption is introduced, a box is opened. Discharging assumptions is achieved by closing a box according to the pattern of its particular proof rule.

Example 1

$$p \wedge \neg q \rightarrow r,\ \neg r,\ p \vdash q$$

1	$p \wedge \neg q \rightarrow r$	premise
2	$\neg r$	premise
3	p	premise
4	$\neg q$	assumption
5	$p \wedge \neg q$	$\wedge$i 3, 4
6	r	$\rightarrow$e 1, 5
7	$\bot$	$\neg$e 6, 2
8	$\neg\neg q$	$\neg$i 4–7
9	q	$\neg\neg$e 8

Example 2

$p \to q \vdash \neg p \lor q$

1	$p \to q$	premise
2	$\neg p \lor p$	LEM
3	$\neg p$	assumption
4	$\neg p \lor q$	$\lor i_1$ 3
5	p	assumption
6	q	$\to e$ 1, 5
7	$\neg p \lor q$	$\lor i_2$ 6
8	$\neg p \lor q$	$\lor e$ 2, 3–4, 5–7

Example 3 (Law of Excluded Middle)

$$\frac{}{\phi \vee \neg\phi} \text{ LEM}$$

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1
5	$\neg\phi$	$\neg\mathrm{i}$ 2–4
6	$\phi \vee \neg\phi$	$\vee\mathrm{i}_2$ 5
7	$\bot$	$\neg\mathrm{e}$ 6, 1
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg\mathrm{i}$ 1–7
9	$\phi \vee \neg\phi$	$\neg\neg\mathrm{e}$ 8

Summary of proof rules

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\,\wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\,\wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\,\wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\,\vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\,\vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\,\vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\,\rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\,\rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\,\neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\,\neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\,\bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\,\neg\neg\mathrm{e}$

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{RAA} \qquad\qquad \frac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

Proof Tips

- First, write down premises at the top of the paper
- Second, write down a conclusion at the bottom of the paper
- Third, look at the structure of the conclusion and try to find compatible proof rules backwardly
  - Pattern matching works, although not all the time.