Propositional Calculus
- Soundness & Completeness of $\mathcal{H}$

Moonzoo Kim
CS Dept. KAIST

moonzoo@cs.kaist.ac.kr

Review

- Goal of logic
  - To check whether given a formula ϕ is valid
  - To prove a given formula ϕ

Sound & Complete

$\vdash \phi$ $\equiv$ $\models \phi$

Syntactic method (proof system $\mathcal{G}$, $\mathcal{H}$, natural deduction,etc)

Semantic method (truth table, etc)

$\equiv$ **Sound & Complete**

$\equiv$ **Sound & Complete**

Semantic tableau

Soundness of $\mathcal{H}$ (1/2)

- Thm 3.34 $\mathcal{H}$ is sound, that is $\vdash$ A then $\vDash$ A
  - Proof is by structural induction
  - We show that
    1. the all three axioms are valid and that
    2. if the premises of MP are valid, so is the conclusion
- Task 1: to prove $\vDash$ Axiom1, $\vDash$ Axiom2, and $\vDash$ Axiom3
  - By showing the semantic tableau of the negated axiom is closed

$$\neg[A \rightarrow (B \rightarrow A)]$$
$$\downarrow$$
$$A, \neg(B \rightarrow A)$$
$$\downarrow$$
$$A, B, \neg A$$
$$\times$$

$$\neg[(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)]$$
$$\downarrow$$
$$\neg B \rightarrow \neg A, \neg(A \rightarrow B)$$
$$\downarrow$$
$$\neg B \rightarrow \neg A, A, \neg B$$
$$\swarrow \qquad \searrow$$
$$\neg\neg B, A, \neg B \qquad \neg A, A, \neg B$$
$$\downarrow \qquad\qquad \times$$
$$B, A, \neg B$$
$$\times$$

Soundness of $\mathcal{H}$ (2/2)

- Task 2: proof by RAA (귀류법)
  - Suppose that MP were not sound.
    - Then there would be a set of formulas {A, A $\rightarrow$ B, B} such that A and A $\rightarrow$ B are valid, but B is not valid

$$\frac{\vdash A \qquad \vdash A \rightarrow B}{\vdash B}$$

    - If B is not valid, there is an interpretation v such that v(B) = F. Since A and A $\rightarrow$ B are valid, for any interpretation, in particular for v, v(A) = v(A $\rightarrow$ B) = T. From this we deduce that v(B) = T contradicting the choice of v

Completeness of $\mathcal{H}$(1/5)

- Thm 3.35 $\mathcal{H}$ is complete, that is, if $\vDash A$ then $\vdash A$
- Any valid formula can be proved in $\mathcal{G}$ (thm 3.8). We will show how a proof in $\mathcal{G}$ can be mechanically transformed into a proof in $\mathcal{H}$
- The exact correspondence is that if the set of formulas U is provable in $\mathcal{G}$ then the single formula $\vee U$ is provable in $\mathcal{H}$
  - A problem is that
    - We can show that $\{\neg p, p\}$ is an axiom in $\mathcal{G}$ then $\vdash p \vee \neg p$ in $\mathcal{H}$ since this is simply Thm 3.10 ($\vdash A \rightarrow A$)
      - Note that $A \vee B$ is an abbreviation for $\neg A \rightarrow B$
      - Similarly $A \wedge B$ is an abbreviation for $\neg (A \rightarrow \neg B)$
    - But if the axiom in $\mathcal{G}$ is $\{q, \neg p, r, p, s\}$, we cannot immediately conclude that $\vdash q \vee \neg p \vee r \vee p \vee s$

Completeness of $\mathcal{H}$(2/5)

- Lem 3.36 If $U' \subseteq U$ and $\vdash \vee U'$ (in $\mathcal{H}$) then $\vdash \vee U$ (in $\mathcal{H}$)
- The proof is by induction using Thm 3.31 through 3.33
  - Suppose we have a proof of $\vee U'$. By repeated application of Thm 3.31, we can transform this into a proof of $\vee U''$, where U” is a permutation of the elements of U.
    - Thm 3.31 Weakening $\vdash A \rightarrow A \vee B$ and $\vdash B \rightarrow A \vee B$
  - Now by repeated applications of the commutativity and associativity of disjunction, we can move the elements of U” to their proper places
    - Thm 3.32 Commutativity rule: $\vdash A \vee B \leftrightarrow B \vee A$
    - Thm 3.33 Associativity rule : $\vdash A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C$

Completeness of $\mathcal{H}$(3/5)

- Completeness proof by induction on the structure of the proof in $\mathcal{G}$
  - We are transforming a proof in $\mathcal{G}$ to a proof in $\mathcal{H}$
- Task 1:
  - If U is an axiom, it contains a pair of complementary literals and $\vdash \neg p \vee p$ can be proved in $\mathcal{H}$. BY Lem 3.36, this may be transformed into a proof of $\vee$ U.
  - Lem 3.36 If $U' \subseteq U$ and $\vdash \vee U'$ (in $\mathcal{H}$) then $\vdash \vee U$ (in $\mathcal{H}$)

Completeness of $\mathcal{H}$(4/5)

- Task 2:
  - The last step in the proof of U in $\mathcal{G}$ is the application of an α or β rule.
    - Case 1: An α rule was used in $\mathcal{G}$ to infer $\dfrac{\vdash U_1 \cup \{A_1, A_2\}}{\vdash U_1 \cup \{A_1 \vee A_2\}}$
      - By the inductive hypothesis, $\vdash (\vee U_1 \vee A_1) \vee A_2$ in $\mathcal{H}$ from which we infer $\vdash \vee U_1 \vee (A_1 \vee A_2)$ by associativity
    - Case 2: An β rule was used in $\mathcal{G}$ to infer

$$\frac{\vdash U_1 \cup \{A_1\} \quad \vdash U_2 \cup \{A_2\}}{\vdash U_1 \cup U_2 \cup \{A_1 \wedge A_2\}}$$

      - By the inductive hypothesis, $\vdash \vee U_1 \vee A_1$ and $\vdash \vee U_2 \vee A_2$ in $\mathcal{H}$. From these, we can find a proof of $\vdash \vee U_1 \vee \vee U_2 \vee (A_1 \wedge A_2)$

Completeness of $\mathcal{H}$(5/5)

- From $\vdash \vee U_1 \vee A_1$ and $\vdash \vee U_2 \vee A_2$ in $\mathcal{H}$, we can find a proof of $\vdash \vee U_1 \vee \vee U_2 \vee (A_1 \wedge A_2)$ as follows:

1. $\vdash \bigvee U_1 \vee A_1$
2. $\vdash \neg \bigvee U_1 \rightarrow A_1$
3. $\vdash A_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$
4. $\vdash \neg \bigvee U_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$
5. $\vdash A_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$
6. $\vdash \bigvee U_2 \vee A_2$
7. $\vdash \neg \bigvee U_2 \rightarrow A_2$
8. $\vdash \neg \bigvee U_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$
9. $\vdash \bigvee U_1 \vee \bigvee U_2 \vee (A_1 \wedge A_2)$

Consistency

- Def 3.38 A set of formulas U is inconsistent iff for some formula A, $U \vdash A$ and $U \vdash \neg A$. U is consistent iff it is not inconsistent
- Thm 3.39 U is inconsistent iff for all A, $U \vdash A$
  - Proof: Let A be an arbitrary formula. Since U is incosistent, for some formula B, $U \vdash B$ and $U \vdash \neg B$.
  - By Thm 3.21 $\vdash B \rightarrow (\neg B \rightarrow A)$. Using MP twice, $U \vdash A$.
- Corollary 3.40 U is consistent iff for some A, $U \nvdash A$
- Thm 3.41 $U \vdash A$ iff $U \cup \{\neg A\}$ is inconsistent

Variants of $\mathcal{H}$ (1/2)

- Variant Hilbert systems almost invariably have MP as the only rule. They differ in the choice of primitive operators and axioms
- $\mathcal{H}$' replace Axiom 3 by
  - Axiom 3' $\vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
- Thm 3.44 $\mathcal{H}$ and $\mathcal{H}$' are equivalent
  - A proof of Axiom 3' in $\mathcal{H}$
  - The other direction?

1.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B$	Assumption
2.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow A$	Assumption
3.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A$	MP 1,2
4.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow \neg A$	Assumption
5.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A \rightarrow B$	Contrapositive 4
6.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash B$	MP 3,5
7.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow B$	Deduction 7
8.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash (\neg B \rightarrow B) \rightarrow B$	Theorem 3.29
9.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash B$	MP 8,9
10.	$\{\neg B \rightarrow \neg A\} \vdash (\neg B \rightarrow A) \rightarrow B$	Deduction 9
11.	$\vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$	Deduction 10

Variants of $\mathcal{H}$ (2/2)

- $\mathcal{H}$" has the same MP rule but a set of axioms as
  - Axiom 1 $\vdash A \lor A \to A$
  - Axiom 2 $\vdash A \to A \lor B$
  - Axiom 3 $\vdash A \lor B \to B \lor A$
  - Axiom 4 $\vdash B \to C \to (A \lor B \to A \lor C)$
  - Note that it is also possible to consider $\lor$ as the primitive binary operator. Then, $\to$ is defined by $\neg A \lor B$.
- Yet another variant of Hilbert system $\mathcal{H}$''' has only one axiom with MP
  - Meredith's axiom
    - $(\{[A \to B) \to (\neg C \to \neg D)] \to C\} \to E) \to [(E \to A) \to (D \to A)]$

Subformula property

- Def 3.48 A deductive system has the subformula property if any formula appearing in a proof of A is either a subformula of A or the negation of a subformula of A
- $\mathcal{G}$ has the subformula property while $\mathcal{H}$ obviously does not since MP 'erase' formulas
  - That is why a proof in $\mathcal{H}$ is harder than a proof in $\mathcal{G}$
- If a deductive system has the subformula property, then mechanical proof may be possible since there exists only
  - there exist only a finite number of subformulas for a finite formula ϕ
  - there exist only a finite number of inference rules

$$\neg p, q, p \qquad\qquad \neg q, q, p$$
$$\searrow \qquad \swarrow$$
$$\neg(p \vee q), q, p$$
$$\downarrow$$
$$\neg(p \vee q), (q \vee p)$$
$$\downarrow$$
$$(p \vee q) \rightarrow (q \vee p)$$

A proof of
$(p \vee q) \rightarrow (q \vee p)$ in $\mathcal{G}$

Automated proof

- One desirable property of a deductive system is to generate an automated/mechanical proof
  - We have decision procedure to check validity of a propositional formula automatically (i.e., truth table and semantic tableau)
    - Note that decision procedure requires knowledge on all interpretations (i.e., infinite number of interpretations in general) which is not feasible except propositional logic
- A deductive proof requires only a finite set of sets of formulas, because a deductive proof system analyzes the target formula only, not its interpretations.
  - Many research works to develop (semi)automated theorem prover
- No obvious connection between the formula and its proof in $\mathcal{H}$ makes a proof in $\mathcal{H}$ difficult (´ no mechanical proof)
  - A human being has to rely on his/her brain to select proper axioms