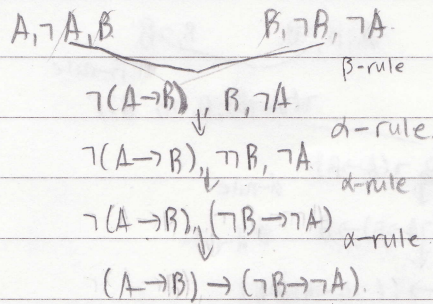


1. $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

in \mathcal{G}



in \mathcal{H}

1. $\{ A \rightarrow B, \neg B, \neg A \} \vdash \neg A$ assumption
2. $\{ A \rightarrow B, \neg B, \neg A \} \vdash A$ double neg 1.
3. $\{ A \rightarrow B, \neg B, \neg A \} \vdash A \rightarrow B$ assumption.
4. $\{ A \rightarrow B, \neg B, \neg A \} \vdash B$ MP. 2, 3
5. $\{ A \rightarrow B, \neg B, \neg A \} \vdash \neg B$ assumption.
6. $\{ A \rightarrow B, \neg B, \neg A \} \vdash \neg B \rightarrow (B \rightarrow \neg B)$ theorem 3.20
7. $\{ A \rightarrow B, \neg B, \neg A \} \vdash B \rightarrow \neg B$ MP. 5, 6
8. $\{ A \rightarrow B, \neg B, \neg A \} \vdash \neg B$ MP 4, 7.
9. $\{ A \rightarrow B, \neg B \} \vdash \neg A \rightarrow \neg B$ deduction 8.
10. $\{ A \rightarrow B, \neg B \} \vdash \neg B \rightarrow \neg A$ contrapositive 9.
11. $\{ A \rightarrow B, \neg B \} \vdash \neg B$ assumption.
12. $\{ A \rightarrow B, \neg B \} \vdash \neg A$ MP 11, 10
13. $\{ A \rightarrow B \} \vdash \neg B \rightarrow \neg A$ deduction 12
14. $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ deduction 13

$$\vdash (A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B)$$

ing

$$\begin{array}{c}
 \neg A, A, B \quad A, B, \neg B \quad \neg A, B, \neg B \quad B, \neg B \\
 \beta\text{-rule} \quad \beta\text{-rule} \\
 \neg(\neg A \rightarrow B), A, B \quad \neg(\neg A \rightarrow B), B, \neg B \\
 \neg(A \rightarrow B), B, \neg(A \rightarrow B) \quad d\text{-rule} \\
 \neg(A \rightarrow B), (\neg A \rightarrow B) \rightarrow B \quad d\text{-rule} \\
 (A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B)
 \end{array}$$

in \mathcal{H}

$$1. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash \neg B \quad \text{assumption}$$

$$2. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash \neg A \rightarrow B \quad \text{assumption}$$

$$3. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash (\neg A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \quad \text{above theorem (I-1)}$$

$$4. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash \neg B \rightarrow \neg A \quad \text{MP } 2, 3$$

$$5. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash \neg \neg A \quad \text{MP } 1, 4$$

$$6. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash A \quad \text{double neg } 5.$$

$$7. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash A \rightarrow B \quad \text{assumption}$$

$$8. \{ A \rightarrow B, \neg A \rightarrow B, \neg B \} \vdash B \quad \text{MP } 6, 7$$

$$9. \{ A \rightarrow B, \neg A \rightarrow B \} \vdash \neg B \rightarrow B \quad \text{deduction } 8.$$

$$10. \{ A \rightarrow B, \neg A \rightarrow B \} \vdash (\neg B \rightarrow B) \rightarrow B \quad \text{Theorem } 3.29$$

$$11. \{ A \rightarrow B, \neg A \rightarrow B \} \vdash B \quad \text{MP } 9, 10$$

$$12. \{ A \rightarrow B \} \vdash (\neg A \rightarrow B) \rightarrow B \quad \text{deduction } 11$$

$$13. \vdash (A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B) \quad \text{deduction } 12.$$

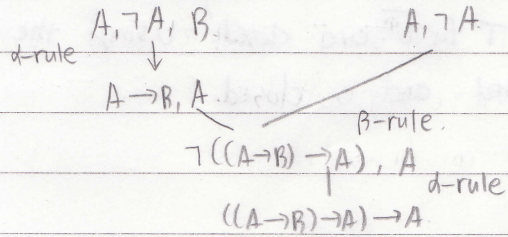
$$(\neg B \wedge (A \rightarrow B)) \rightarrow \neg A$$

$$\neg(\neg B \rightarrow \neg(A \rightarrow B)) \rightarrow \neg A$$

Date

No.

$$\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$$

in \mathcal{G} in \mathcal{H}

1. $\{(A \rightarrow B) \rightarrow A, \neg A\} \vdash \neg A$ assumption
2. $\{(A \rightarrow B) \rightarrow A, \neg A\} \vdash \neg A \rightarrow (A \rightarrow B)$ Theorem 3.20
3. $\{(A \rightarrow B) \rightarrow A, \neg A\} \vdash (A \rightarrow B) \wedge \neg A$ MP 1, 2
4. $\{(A \rightarrow B) \rightarrow A, \neg A\} \vdash (A \rightarrow B) \rightarrow A$ assumption
5. $\{(A \rightarrow B) \rightarrow A, \neg A\} \vdash A \rightarrow B$ MP 3, 4
6. $\{(A \rightarrow B) \rightarrow A\} \vdash \neg A \rightarrow A$ deduction 5
7. $\{(A \rightarrow B) \rightarrow A\} \vdash (A \rightarrow A) \rightarrow A$ Theorem 3.29
8. $\{(A \rightarrow B) \rightarrow A\} \vdash A$ MP 6, 7
9. $\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$ deduction 8

2. Prove that if $\mathcal{H} \cup \text{ing}$ then there is a closed semantic tableau for \bar{U}

We prove that there is a closed semantic tableau for \bar{U} by induction on h , the height of a proof tree. If $h=0$, then U is an axiom, containing a complementary pair $\{p, \neg p\}$, that is, $\bar{U} = \bar{U}_0 \cup \{p, \neg p\}$. Semantic tableau rules cannot decompose literals, so $\{p, \neg p\}$ is in all leaf nodes, so \mathcal{T} for \bar{U} is closed.

If $h>0$, then some Gentzen α - or β -rule was used at the root of a proof tree on a formula $A \in U$, that is, $U = U_0 \cup \{A\}$.

1) Case 1: A Gentzen α -rule was used on $A = A_1 \vee A_2$ to produce the node

n labeled $U' = U_0 \cup \{A_1, A_2\}$. The subtree rooted at n is a proof tree for U'

so by the inductive hypothesis, \mathcal{T} for $\bar{U}' (= \bar{U}_0 \cup \{\neg A_1, \neg A_2\})$ is closed. Using the semantic tableau α -rule, \mathcal{T} for $\bar{U} (= \bar{U}_0 \cup \{\neg(A_1 \vee A_2)\})$ can be constructed and is closed.

Case 2: A Gortzen β -rule was used on $A = A_1 \wedge A_2$ to produce the nodes n' and n'' labeled $U' = U_0 \cup \{A_1\}$ and $U'' = U_0 \cup \{A_2\}$, respectively. By the inductive hypothesis; T' for U' and T'' for U'' are closed. Using the semantic tableau β -rule T for U can be constructed, and is closed.

3.

$$\frac{\frac{\frac{\vdash A \rightarrow B}{\vdash \neg B} \quad \vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)}{\vdash \neg B \rightarrow \neg A} \quad \vdash \neg B}{\vdash \neg A} \beta$$

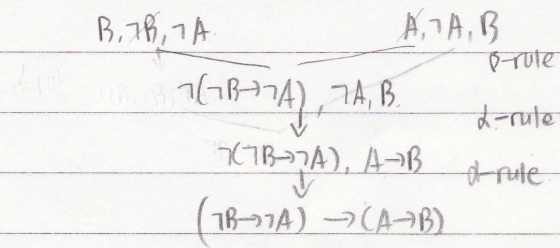
4. 1. $\vdash A \rightarrow (B \rightarrow A)$

$$\begin{array}{l} A, \neg A, \neg B \\ \downarrow \text{d-rule} \\ \neg A, (B \rightarrow A) \\ \downarrow \text{d-rule} \\ A \rightarrow (B \rightarrow A) \end{array}$$

2. $\vdash ((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$

$$\begin{array}{l} A, \neg A, B, C \quad B, \neg B, \neg A, C \\ \downarrow \beta\text{-rule} \\ \neg A, B, C, \neg(A \rightarrow B) \quad (A, C, \neg(A \rightarrow B)) \\ \downarrow \beta\text{-rule} \\ \neg(B \rightarrow C), \neg(A \rightarrow B), \neg A, C \quad A, \neg A \\ \downarrow \beta\text{-rule} \\ \neg(A \rightarrow B), \neg(A \rightarrow (B \rightarrow C)), \neg A, C \\ \downarrow \text{d-rule} \\ \neg(A \rightarrow B), \neg(A \rightarrow (B \rightarrow C)), (A \rightarrow C) \\ \downarrow \text{d-rule} \\ ((A \rightarrow B) \rightarrow (A \rightarrow C)), \neg(A \rightarrow (B \rightarrow C)) \\ \downarrow \text{d-rule} \\ (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \end{array}$$

3. $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$



5. $\vdash (\neg A \rightarrow A) \rightarrow A$ in H

1. $\{ \neg A \rightarrow A, \neg A \} \vdash \neg A$ assumption

2. $\{ \neg A \rightarrow A, \neg A \} \vdash \neg A \rightarrow A$ assumption

3. $\{ \neg A \rightarrow A, \neg A \} \vdash A$ MP 1, 2

4. $\{ \neg A \rightarrow A, \neg A \} \vdash A \rightarrow (\neg A \rightarrow \text{false})$ theorem 3.21

5. $\{ \neg A \rightarrow A, \neg A \} \vdash \neg A \rightarrow \text{false}$ MP 3, 4

6. $\{ \neg A \rightarrow A, \neg A \} \vdash \text{false}$ MP 1, 5

7. $\{ \neg A \rightarrow A \} \vdash \neg A \rightarrow \text{false}$ deduction 6.

8. $\{ \neg A \rightarrow A \} \vdash A$ RAA 7.

9. $\vdash (\neg A \rightarrow A) \rightarrow A$ deduction 8