# Solution of Homework 3

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If you have any question related to this homework, please send me
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## 1

Preticate logics are (a), (b), (d), (f), (g).
For wrong strings, (c) is not a predicate but a term because $f$ is a function.
In (e) and (h), $B(m)$ and $B(x)$ are not a term because $B$ is a predicate.

## 2

(a) $\forall p.(P(x) \rightarrow A(m, x))$
(b) $\exists p.(P(x) \wedge A(x, m))$
(c) $A(m, m)$
(d) $\neg\exists x.(S(x) \wedge (\forall y.B(x, y)))$
(e) $\neg\exists l.(L(l) \wedge \forall s.(S(s) \rightarrow B(s, l)))$
(f) $\neg\exists l.(L(l) \wedge \neg\exists s.(S(s) \wedge B(s, l)))$

## 3

Predicate logics are (c) and (f).
Parse tree of (c) is like below.
Parse tree of (f) is like below. For wrong strings,
In (a), (b), and (d), the number of arguments of $P$ is incorrect.
In (e), $g$ is not a predicate but a function. Therefore $g(x, y)$ can not be used as a predicate formula.

## 4

(a) $\forall x.(F(x) \rightarrow \exists y.(Q(y, x)))$
(b) $B(j, c) \rightarrow \neg\exists y.(F(y) \wedge L(j, y))$
(c) $\exists x.(F(x) \wedge B(c, x) \wedge B(x, j))$

## 5

$Red(x) : x\ is\ red.$
$Box(x) : x\ is\ in\ the\ box.$

(a) $\forall x.(Red(x) \rightarrow Box(x))$
(b) $\forall x.(Box(x) \rightarrow Red(x))$

$Animal(x)$ : $x$ is an animal.
$Cat(x)$ : $x$ is a cat.
$Dog(x)$ : $x$ is a dog.

(c)$\neg\exists x.(Animal(x) \wedge Cat(x) \wedge Dog(x))$

$Boy(x)$ : $x$ is a boy.
$Prize(y)$ : $y$ is a prize.
$Won(x, y)$ : $x$ wons $y$.

(d) $\forall x.(Prize(x) \rightarrow \exists y.(Boy(y) \wedge Won(y, x)))$
(e) $\exists x.(Boy(x) \wedge \forall y.Won(x, y))$

## 6

(a) $\forall x.\exists y.M(y, x)$

(b) $\forall x.(\exists y.M(y, x) \wedge \exists z.F(z, x))$

(c) $\forall x.((\exists y.M(y, x)) \rightarrow (\exists z.F(z, x)))$

(d) $\exists x.\exists y.(F(Ed, y) \wedge F(y, x))$

(e) $\forall x.((\exists y.F(x, y)) \rightarrow \exists z.H(x, z))$

(f) $\forall x.\exists y.(H(x, y) \rightarrow \exists c.(F(x, c) \wedge M(y, c)))$
If you clearly define the meaning of (e) and (f) and specify them correctly, then it is okay.

(g) $\neg\exists u, m, f, x.(F(f, x) \wedge B(f, u) \wedge M(m, x) \wedge S(m, u))$

(h) $\forall x.\exists y.(B(x, y) \rightarrow \exists f.(F(f, x) \wedge F(f, y)))$

(i) $\neg\exists g, m, x.(M(m, x) \wedge M(g, m) \wedge \exists a.F(g, a))$

(j) $H(Ed, Patsy)$

(k) $\exists h.(H(h, Monique) \wedge B(h, Carl))$

## 7

$Visit(x, y)$ : $x$ visited $y$.
$Love(x, y)$ : $x$ loves $y$.

(a) $\forall x.(visit(x, NewOrleans) \rightarrow Love(x, NewOrleans))$

$Play(x, y)$ : $x$ plays $y$.
$LiveIn(x, y)$ : $x$ lives in $y$.
$Like(x, y)$ : $x$ likes $y$.

(b) $\exists x.(Play(x, Trumphet) \wedge LiveIn(x, NewOrleans) \wedge \neg Like(x, crawfishetouffee)$.

$BornIn(x, y)$ : $x$ born in $y$.
$Better(x, y, z)$ : $x$ plays $y$ better than $z$.

(c) $\exists p, q.(p \neq q \wedge BornIn(p, NewOrleans) \wedge BornIn(q, NewOrleans) \wedge Play(p, Saxophone) \wedge Play(q, Saxophone) \wedge \forall x.(LiveIn(x, NewYork) \wedge Play(x, Saxophone) \wedge Better(p, Saxophone, x) \wedge Better(q, Saxophone, x)))$

## 8

$M = (\{0, 1\}, \{\{Q\}, \{g\}\}, \{\})$
$where\ g = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 0)\}\ and\ Q = \{(0, 0, 0)\}$

$M^{'} = (\{0, 1\}, \{\{Q\}, \{g\}\}, \{\})$
$where\ g = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}\ and\ Q = \{(0, 0, 0)\}$

and $l(x) = 1$, $l(y) = 1$, $l(z) = 0$, $l^{'}(x) = 1$, $l^{'}(y) = 1$, $l^{'}(z) = 0$.

## 9

(a) For any interpretation of $x$, $M \models_l \phi$ if $l(y) = x + 1$ and $l(z) = x$.
Therefore $M \models \phi$.
(b) For any interpretation of $x$, $M^{'} \models_l \phi$ if $l(y) = 2 * x$ and $l(z) = x$.

Therefore $M \models \phi$.
(c) For any interpretation of $x$, $M^{''} \models_l \phi$ if $l(y) = x+1$ and $l(z) = x$.
Therefore $M \models \phi$.

## 10

$M = (N, \{\neq\}, \{\})$
$M^{'} = (N, \{\equiv\}, \{\})$

## 11

$\phi = \forall x.(\exists y.P(x,y) \wedge (\exists z.(P(z,x) \rightarrow \forall y.P(x,y))))$.

A model of choice is $M = \{\{0,1,2\}, \{P\}, \{\}\}$ where $P^M = \{(0,1),(1,1),(2,1)\}$.

To prove $M \models \phi$,

$l[x \mapsto 0], M \models_l \phi$ if $M \models_l \exists y.P(0,y) \wedge (\exists z.(P(z,0) \rightarrow \forall y.P(0,y)))$
$l[x \mapsto 0, y \mapsto 0], M \not\models_l P(0,0)$ so $M \not\models_l \phi$.
$l[x \mapsto 0, y \mapsto 1], M \models_l \phi$ if $M \models_l \exists z.(P(z,0) \rightarrow \forall y.P(0,y)$.
$l[x \mapsto 0, y \mapsto 1, z \mapsto 0], M \models_l P(0,0) \rightarrow \forall y.P(0,y)$.
$l[x \mapsto 1, y \mapsto 0], M \not\models P(1,0)$ so $M \not\models_l \phi$.
$l[x \mapsto 1, y \mapsto 1], M \models_1 \exists z.(P(z,1) \rightarrow \forall y.P(1,y)$
$l[x \mapsto 1, y \mapsto 1, z \mapsto 0], M \models_l \phi if M \models_l \forall y.P(1,y)$
$l[x \mapsto 1, y \mapsto 1, z \mapsto 0, y' \mapsto 0], P(1,0) = false$.
$l[x \mapsto 1, y \mapsto 1, z \mapsto 1], M \models_l \phi if M \models_l \forall y.P(1,y)$
$l[x \mapsto 1, y \mapsto 1, z \mapsto 1, y' \mapsto 0], P(1,0) = false$.
$l[x \mapsto 1, y \mapsto 1, z \mapsto 2], M \models_l \phi if M \models_l \forall y.P(1,y)$
$l[x \mapsto 1, y \mapsto 1, z \mapsto 2, y' \mapsto 0], P(1,0) = false$.
$l[x \mapsto 1, y \mapsto 2], M \not\models P(1,2)$ so $M \not\models_l \phi$.

Therefore, $M \not\models \phi$.

## 12

(a) 12
(b) 69
(c) 3728

## 13

(a) $M \models \phi$ does not hold for an interpretation where $l(x) = b$ and $l(y) = a$.
(b) $M \models \phi$ holds for any intepretation. If $l(x) = b$ then $l(y)$ only can be $c$ and $\exists z.R(c, z)$ holds for the model. If $l(x) = a$ then $l(y)$ only can be $b$ and $\exists z.R(b, z)$ holds for the model. Finally, if $l(x) = c$ then $l(y)$ only can be $b$ and $\exists z.R(b, z)$ holds for the model.

## 14

$(a)\ M = \{\{0,1\}, \{S, P\}, \{\}\}\ where\ S^M = \{(0,1),(1,0)\}\ and\ P^M = \{0\}.$
$(b)\ M = \{\{0,1,2\}, \{S\}, \{\}\}\ where\ S^M = \{(0,1),(1,2),(0,2)\}.$
$(c)\ M = \{\{0,1,2\}, \{R, Q, P\}, \{\}\}\ where\ R^M = \{0\}\ , Q^M = \{0,1\}\ and\ P^M = \{2\}.$
$(d)\ M = \{\{0,1\}, \{S\}, \{\}\}\ where\ S^M = \{(0,0),(1,1)\}.$

## 15

(a) Not Valid. $M = \{\{0,1\}, \{P\} \{\}\}$ where $P^M = \{0\}$.
(b) Not Valid. $M = \{\{0,1\}, \{P, Q\} \{\}\}$ where $P^M = \{0,1\}$ and $Q^M = \{0\}$.
(c) Valid.
(d) Valid.

## 16

$(a)$ $M = \{\{0,1\}, \{S\}, \{\}\}$ where $S^M = \{(0,0),(1,1)\}$.

$(b)$If $\forall x.P(x) = T$, then it is obvious that $\exists y.P(y) = T$ and therefore the formula is true. Otherwise, the formula is true because $false \to \exists y.P(y)$ is always true..

$(c)and(d)$by logical equivalence(p.108 in textbook).

$(e)$ $M = \{\{0,1,2\}, \{S\}, \{\}\}$ where $S^M = \{(0,2)\}$.

$(f)$ $M = \{\{0,1\}, \{S\}, \{\}\}$ where $S^M = \{(0,0)\}$.

## 17

$U_1$ is a poset which has either common lowest element or common highest element( e.g. $U_1 = \{(a,b),(b,c),(c,d),(e,f),(f,g),(g,c)\}$).

$U_2$ is a poset which has neither common lowest element nor common highest element. So the set can be decomposed into independent element sets( e.g. $U_2 = \{(a,b),(b,c),(c,d),(e,f),(f,g),(g,h)\}$).

## 18

M=$(D, \{\{G, =\}, \{op\}\}, \{\})$

- *Associativity*
  $\forall a,b,c.(G(a) \wedge G(b) \wedge G(c) \wedge (= (op(a, op(b,c)), op(op(a,b),c))))$
- *Identity element*
  $\exists i.(G(i) \wedge \forall a.(G(a) \rightarrow = (op(i,x), op(x,i))))$
- *Inverse element*
  $\exists i.((G(i) \wedge \forall a.(G(a) \rightarrow = (op(i,x), op(x,i)))) \wedge \forall x.(G(x) \rightarrow \exists y.(G(y) \wedge = (i, op(x,y)))))$

## 19

$I_1 = 0 < 1 < 2 < 3....$ and $I_2 = 0 \triangle 2 \triangle 4 \triangle 6 ... \triangle 1 \triangle 3 \triangle 5...$
$\phi_{op} = \forall x.((0\ op\ x) \rightarrow \forall y.\neg(y\ op\ x))$ and $\neg\phi_{op} = \exists x.((0\ op\ x) \wedge \exists y.(y\ op\ x))$.
Then, $I_1 \models \phi_{<}$and , $I_2 \not\models \phi_{\triangle}$.